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AN ANALYTICAL STUDY OF THE LOGARITHMIC MEAN TEMPERATURE DIFFERENCE

ESTUDIO ANALITICO DE LA MEDIA LOGARITMICA DE DIFERENCIA DE TEMPERATURAS

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Abstract

The logarithmic mean temperature difference (LMTD) has caused inconveniences in several applications like equation-oriented flow sheeting programs. Such inconveniences have arisen from its indeterminate form. This is a consequence of the incomplete model derivation generally developed in the textbooks. Heat exchanger dynamic analysis and control synthesis through lumped-parameter models using the LMTD as *driving force* (fluid mean temperature difference) may suffer from such inconsistencies too. This paper is devoted to give a solution to such inconveniences by providing a formal mathematical treatment of the LMTD. First, a complete derivation is restated resulting in a complete well-defined expression. Then, several interesting analytical properties of the resulting expression, like continuous differentiability on its domain, are proved. The usefulness of the results is highlighted throughout the text.

Keywords: logarithmic mean temperature difference, heat exchangers.

Resumen

La media logarítmica de diferencia de temperaturas (LMTD, por sus siglas en inglés) ha causado inconveniencias en diversas aplicaciones tales como ciertos programas de simulación de procesos. Su forma indeterminada es una de las principales causas de tales inconveniencias. Tal indeterminación es una consecuencia del procedimiento incompleto que generalmente se desarrolla para su obtención en los libros de texto. El análisis dinámico y el diseño de control de intercambiadores de calor a través de modelos de parámetros agrupados que usan la LMTD como *fuerza conductora* (de intercambio de calor, *i.e.* diferencia promedio de temperatura entre los fluidos) pueden también ser víctimas de tales inconsistencias. Este trabajo está dedicado a dar una solución a tales inconveniencias a través de un análisis matemático formal de la LMTD. Primero, un procedimiento completo para su obtención es desarrollado, dando como resultado una expresión completa bien definida. Posteriormente, diversas propiedades analíticas de la expresión resultante, tales como la continuidad y la diferenciabilidad en todo su dominio, son probadas. La utilidad de los resultados es comentada a lo largo del texto.

Palabras clave: media logarítmica de diferencia de temperatura, intercambiadores de calor.

1. Introduction

The logarithmic mean temperature difference (LMTD)

$$\Delta T_{\ell} = \frac{\Delta T_2 - \Delta T_1}{\ln \frac{\Delta T_2}{\Delta T_1}} \qquad (1)$$

plays an important role in theoretical and practical aspects of heat exchangers. It is involved in their design (Reimann, 1986; Incropera and DeWitt, 1990); performance calculation (Incropera and DeWitt, 1990; Holman, 1997); steady-state analysis (Kern, 1950; Mathisen, 1994); dynamic modelling (Reimann, 1986; Steiner, 1989; Steiner, 1987), simulation (Papastratos, *et al.*; Zeghal *et al.*, 1991), and characterization (Zavala-Río *et al.*, 2003, Zavala-Río and Santiesteban-Cos, 2004); (closed-loop) stability-limit analysis (Khambanonda *et al.*, 1991; Khambanonda *et*

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al., 1990); and control synthesis (Alsop and Edgar, 1989; Malleswararao and Chidambaram, 1992). For the sake of simplicity, the use of less involved expressions, like the arithmetic model

$$\Delta T_a = \frac{\Delta T_1 + \Delta T_2}{2} \tag{2}$$

or the geometric one

$$\Delta T_g = \sqrt{\Delta T_1 \Delta T_2} \tag{3}$$

is sometimes preferred to approximate the mean temperature difference along the exchanger. However, among these expressions, it is just the LMTD that takes into account the exponential distribution of the fluid temperatures over the tube length in stationary conditions (Incropera and DeWitt, 1990). Consequently, more appropriate steady-state values are computed using (1) than (2) or (3). Nevertheless, the LMTD may cause inconvenience if it is taken simply as shown in (1). Paterson (1984), for instance, points out that the iterative equation-solving procedures performed by equation-oriented flow sheeting programs, commonly take starting values that involve the equality of stream temperatures, and hence zero temperature differences where (1) reduces to an indeterminate form; moreover, taking $\Delta T_1 = \Delta T_2$, the derivatives of ΔT_ℓ with respect to ΔT_1 and ΔT_2, needed in the Newton iterative solution of the equations, are undefined. On the other hand, heat exchanger dynamic analysis and control design may suffer from such inconveniences too. In (Alsop and Edgar, 1989; Malleswararao and Chidambaram, 1992), for instance, the authors take the following 2nd-order lumped-parameter model

$$\begin{aligned} \frac{dT_{co}}{dt} &= \frac{2}{M_c}\left[F_c\,(T_{ci} - T_{co}) + \frac{UA}{C_{pc}}\,\Delta T\right] \\ \frac{dT_{ho}}{dt} &= \frac{2}{M_h}\left[F_h\,(T_{hi} - T_{ho}) + \frac{UA}{C_{ph}}\,\Delta T\right] \end{aligned} \tag{4}$$

and define ΔT as in (1) to design control schemes for countercurrent heat exchangers. Using F_c as control input, they propose an algorithm to stabilize T_{ho} at a desired value. Nevertheless, under such a modelling, existence of solutions going through (or starting at) state values such that $\Delta T_1 = \Delta T_2$ is undefined due to the undeterminate form of (1). Moreover, the proposed control schemes make use of the derivatives of ΔT_ℓ with respect to ΔT_1 and ΔT_2 which, as above mentioned, are undefined when $\Delta T_1 = \Delta T_2$ (and it is not clear in such works how to handle such an inconsistency). Thus, well posedness of the LMTD turns out to be important.

In order to deal with the above mentioned inconveniences, some authors state that $\Delta T_\ell \to \Delta T_a$ as $\Delta T_2 - \Delta T_1 \to 0$ (see *e.g.* (Kern, 1959; Gardner and Taborek, 1977)), or simply suggest to replace ΔT_ℓ by ΔT_a when $\Delta T_1 = \Delta T_2$ (see *e.g.* (Faires, 1957)), and others claim further that $\frac{\partial \Delta T_\ell}{\partial \Delta T_i} \to \frac{1}{2}$, i = 1, 2, as $\Delta T_2 - \Delta T_1 \to 0$ (see *e.g.* (Paterson, 1984)). However, a mathematical proof of such assertions still seems to be lacking. Some others suggest to replace (1) by an infinite power series expansion that yields ΔT_a when $\Delta T_1 = \Delta T_2$ (see *e.g.* (Steiner, 1989)). Unfortunately, performing calculations through infinite power series imply the consideration of an infinite number of arithmetical operations, while through a truncated version of the series, accuracy is lost. Other authors have proposed well-defined replacement expressions that approximate ΔT_ℓ over an acceptable range of ΔT_1 and ΔT_2 (Paterson, 1984; Underwood, 1970; Chen, 1987). Paterson (1984), for example, proposes

$$\Delta T_{\ell P} = \frac{\Delta T_a + 2\Delta T_g}{3} \approx \Delta T_\ell \tag{5}$$

while

$$\Delta T_{\ell U}^{1/3} = \frac{\Delta T_1^{1/3} + \Delta T_2^{1/3}}{2} \approx \Delta T_\ell^{1/3} \qquad (6)$$

is proposed by Underwood (1970), and

$$\Delta T_{\ell C1} = \Delta T_a^{1/3} \Delta T_g^{2/3} y \approx \Delta T_\ell \qquad (7)$$

and

$$\Delta T_{\ell C2}^{0.3275} = \frac{\Delta T_1^{0.3275} + \Delta T_2^{0.3275}}{2} \approx \Delta T_\ell^{0.3725} \qquad (8)$$

(the latter being a refined modification of Underwood's approximation) by Chen (1987) (a comparison study of these four approximations is presented in (Chen, 1987)).

This work is devoted to provide a formal mathematical treatment of the LMTD, the results of which are intended to give a solution of the above mentioned inconveniences. First, we show that the indeterminate form of (1) is a consequence of the incomplete derivation of the LMTD generally presented in the textbooks (see for instance (Incropera and DeWitt, 1990; Holman, 1997; Kern, 1950; Faires, 1957; McAdams, 1954; Walas, 1991)). By restating a complete derivation, an expression is gotten being equivalent to (1) when $\Delta T_1 \neq \Delta T_2$ and well-defined when $\Delta T_1 = \Delta T_2$. Consequently, approximating the LMTD through replacement expressions may henceforth be avoided. Second, analytical properties of the resulting *complete expression*, such as continuity and differentiability, are proved for every physically reasonable combination of values of ΔT_1 and ΔT_2, including those where $\Delta T_1 = \Delta T_2$. Heat exchanger lumped-parameter dynamic models (like (4)) using the LMTD, and control schemes using its derivative with respect to ΔT_1 and ΔT_2, will now be mathematically coherent through the use of such a complete expression.

The work is organized as follows: Section 2 states the notation adopted in the present study. In Section 3, the complete derivation of the LMTD is developed. Analytical properties of the resulting model are stated in Section 4. Finally, conclusions are given in Section 5.

2. Nomenclature and notation

The following nomenclature is defined for its use throughout this work:

F	mass flow rate
Cp	specific heat
M	total mass inside the tube
U	overall heat transfer coefficient
A	heat transfer surface area
T	temperature
ΔT	temperature difference
Q	rate of heat transfer
t	time
R	set of real numbers
R^2	set of 2-tuples $(x_i)_{i=1,2}$ with $x_i \in R$
R_+	set of positive real numbers
R_+^2	set of 2-tuples $(x_i)_{i=1,2}$ with $x_i \in R$

Subscripts:

c	cold
h	hot
i	inlet
o	outlet

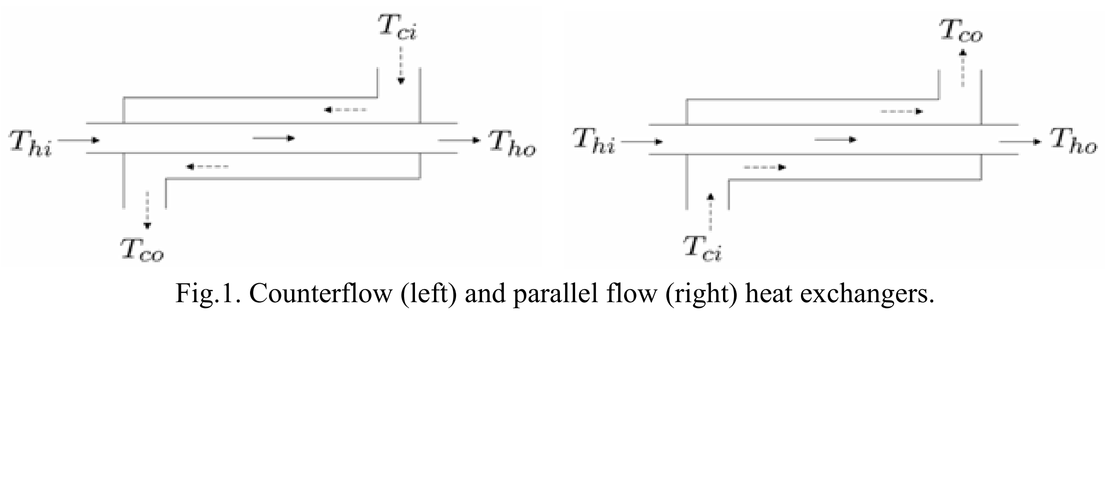

Fig.1. Counterflow (left) and parallel flow (right) heat exchangers.

In particular, Q_h, Q_c, and Q represent the rates of heat convected through the hot and cold fluid tubes and that transferred from the hot to the cold fluid respectively. The variables T_h and T_c generically denote the temperature of the hot and cold fluids, respectively, at any point along the exchanger, and $\Delta T = T_h - T_c$. Furthermore, let ΔT_1 and ΔT_2 represent the temperature difference at each terminal side of the heat exchanger, *i.e.* (see Fig. 1)

$$\Delta T_1 = \begin{cases} T_{hi} - T_{co} & \text{if counterflow} \\ T_{hi} - T_{ci} & \text{if parallel flow} \end{cases} \qquad (9)$$

and

$$\Delta T_2 = \begin{cases} T_{ho} - T_{ci} & \text{if counterflow} \\ T_{ho} - T_{co} & \text{if parallel flow} \end{cases} \qquad (10)$$

taken in this work as conventional definitions. Notice that under such conventional definitions, ΔT_1 and ΔT_2 shall be considered positive. To differentiate from ΔT_ℓ in (1), we denote ΔT_L the well-posed logarithmic mean temperature difference to be derived in the following section. ΔT_a still stands (as in the previous section) for the arithmetic model. Consider the sets B, C, and D with $B \subset C$, and a function $f : C \to D$. We denote $f|_B$ the restriction of f to B, *i.e.* $f|_B : B \to D : x \mapsto f|_B(x) = f(x), \forall x \in B$. The boundary of a subset, say B, is represented as ∂B.

3. Complete derivation of the LMTD

We recall that the present derivation assumes that C_{pc}, C_{ph}, and U, are considered flow and temperature invariant, that is, their value is considered to be constant throughout the exchanger; see for instance (Incropera and DeWitt, 1990; Holman, 1997; Kern, 1950; Faires, 1957; McAdams, 1954; Wallas, 1991); for a longer (plus exhaustive) list of the assumptions involved, see for example (Kern, 1950; Incropera and DeWitt, 1990). Furthermore, the following developments take into account both flow configurations of heat exchangers simultaneously through an auxiliary parameter α. Any expression where α is not present is valid for both flow configurations. In those where α appears, such parameter determines the configuration they are valid for in the following way

$$\alpha = \begin{cases} 1 & \text{if counterflow} \\ -1 & \text{if parallel flow} \end{cases} \qquad (11)$$

At any point along the exchanger, the heat transfer equations are

$$dQ_h = F_h C_{ph} dT_h \qquad (12)$$

$$dQ_c = F_c C_{pc} dT_c \qquad (13)$$

$$dQ_h = \alpha dQ_c = -dQ \qquad (14)$$

$$dQ = U \Delta T dA \qquad (15)$$

From the constant physical property assumption, and considering a mean temperature difference, ΔT_L, throughout the exchanger, we have, after integration over the tube length,

$$Q_h = F_h C_{ph}(T_{ho} - T_{hi}) \qquad (16)$$

$$Q_c = F_c C_{pc}(T_{ci} - T_{co}) \qquad (17)$$

$$Q_h = \alpha Q_c = -Q \qquad (18)$$

$$Q = UA\Delta T_L \qquad (19)$$

The next developments follow

$$\begin{aligned} d\Delta T &= dT_h - dT_c \\ &= \frac{dQ_h}{F_h C_{ph}} - \frac{dQ_c}{F_c C_{pc}} && \text{(from (12) and (13))} \\ &= dQ\left(\frac{1}{\alpha F_c C_{pc}} - \frac{1}{F_h C_{ph}}\right) && \text{(from (14))} \end{aligned}$$

and, from (15), we get

$$d\Delta T = U\Delta T\left(\frac{1}{\alpha F_c C_{pc}} - \frac{1}{F_h C_{ph}}\right)dA \qquad (20)$$

At this point, we consider two possible situations:

The general case. We begin by assuming that

$$\Delta T\left(\frac{1}{\alpha F_c C_{pc}} - \frac{1}{F_h C_{ph}}\right) \neq 0 \qquad (21)$$

Let us note that this is the situation that is most often found in physical heat exchangers[1]. Then, from (20), we have

$$\frac{d\Delta T}{\Delta T} = U\left(\frac{1}{\alpha F_c C_{pc}} - \frac{1}{F_h C_{ph}}\right)dA$$

yielding

$$ln\frac{\Delta T_2}{\Delta T_1} = UA\left(\frac{1}{\alpha F_c C_{pc}} - \frac{1}{F_h C_{ph}}\right) \qquad (22)$$

after integration over the tube length. Notice that since the right-hand side of (22) is a non-zero scalar, ΔT_1 and ΔT_2 (in the left-hand side) must be such that $ln(\Delta T_2 / \Delta T_1)$ be a non-zero scalar too, which is satisfied if and only if $\Delta T_1 \neq \Delta T_2$. Then, one sees from these developments that

$$(21) \Leftrightarrow \Delta T_1 \neq \Delta T_2 \qquad (23)$$

Now, notice that

$$\frac{1}{\alpha F_c C_{pc}} - \frac{1}{F_h C_{ph}} = \frac{T_{ci} - T_{co}}{Q_c} - \frac{T_{ho} - T_{hi}}{Q_h} \qquad \text{(from (16) and (17))}$$

$$= \frac{T_{ho} - T_{hi}}{Q} - \frac{\alpha\left(T_{ci} - T_{co}\right)}{Q} \qquad \text{(from (18))}$$

$$= \frac{\Delta T_2 - \Delta T_1}{Q} \qquad \text{(from (9), (10), and (11))}$$

which can be substituted into (22) to get

$$ln\frac{\Delta T_2}{\Delta T_1} = UA\frac{\Delta T_2 - \Delta T_1}{Q}$$

From this expression, we get

$$Q = UA\frac{\Delta T_2 - \Delta T_1}{ln\frac{\Delta T_2}{\Delta T_1}}$$

which, compared to (19) and taking into account (23), shows that

$$\Delta T_L = \frac{\Delta T_2 - \Delta T_1}{\ln\frac{\Delta T_2}{\Delta T_1}} = \Delta T_\ell \quad \text{if } \Delta T_1 \neq \Delta T_2 \qquad (24)$$

The special case. Let us now suppose that

$$\Delta T\left(\frac{1}{\alpha F_c C_{pc}} - \frac{1}{F_h C_{ph}}\right) = 0 \qquad (25)$$

Although this situation is seldom found in practice[2], it is considered here since its development is needed for the well-posedness of the LMTD. From (20), we have

$$d\Delta T = 0 \qquad (26)$$

implying that the temperature difference $\Delta T = T_h - T_c$ has a constant value throughout the exchanger. Hence, after integration of (26) and (15) over the tube length, we have $\Delta T_2 = \Delta T_1 = \Delta T$ (equivalent to (25)) and

[1] This may explain why it is in fact the case that is generally developed in the textbooks (for a specific flow configuration), implicitly taking (21) as a fact (which is actually the origin of the inconsistency problem of (1)).

[2] This may explain why this case is apparently never taken into account; it thus constitutes the part of the derivation that is generally lacking in the textbooks.

$Q = UA\Delta T$. From these expressions, comparing the latter to (19), we have

$$\Delta T_L = \Delta T \quad \text{if} \quad \Delta T_2 = \Delta T_1 = \Delta T \tag{27}$$

The well-posed LMTD. Finally, from the results gotten in both, the general and special cases, i.e. from (24) and (27), we get

$$\Delta T_L = \begin{cases} \dfrac{\Delta T_2 - \Delta T_1}{ln \dfrac{\Delta T_2}{\Delta T_1}} & \text{if } \Delta T_1 \neq \Delta T_2 \\ \Delta T & \text{if } \Delta T_2 = \Delta T_1 = \Delta T \end{cases} \tag{28}$$

4. Analytical properties

In this section we consider the mean temperature difference a bivariable function, whether we refer to the logarithmic model in (28), $\Delta T_L(\Delta T_1, \Delta T_2)$, or the arithmetic one in (2), $\Delta T_a(\Delta T_1,\Delta T_2)$. Furthermore, just the cases where heat is transferred between fluids are considered of practical interest in the present section. Hence, for analysis purposes, we choose to disregard the no-heat-transfer trivial case $\Delta T_2 = \Delta T_1 = 0$. Under such perspective, R_+^2 is considered to constitute the domain of ΔT_L. We begin by stating a useful equivalent expression.

Lemma 1. *Let*

$$L(\Delta T_1, \Delta T_2) \triangleq 1 + \sum_{i=1}^{\infty} \frac{1}{2i+1} \left(\frac{\Delta T_2 - \Delta T_1}{\Delta T_2 + \Delta T_1} \right)^{2i} \tag{29}$$

for all (ΔT_1, ΔT_2) *such that* $\Delta T_1 + \Delta T_2 \neq 0$. *Then*

$$\Delta T_L(\Delta T_1, \Delta T_2) \equiv \frac{\Delta T_a(\Delta T_1, \Delta T_2)}{L(\Delta T_1, \Delta T_2)} \tag{30}$$

$\forall (\Delta T_1, \Delta T_2) \in R_+^2$.

Proof. We divide the proof in two parts:

1) $\Delta T_1 \neq \Delta T_2$. From Formula 4.1.27 in[3] (Abramowitz and Stegun, 1972), we have

$$ln \frac{\Delta T_2}{\Delta T_1} = 2 \sum_{i=0}^{\infty} \frac{1}{2i+1} \left(\frac{\Delta T_2 - \Delta T_1}{\Delta T_2 + \Delta T_1} \right)^{2i+1}$$

$\forall (\Delta T_1, \Delta T_2) \in R_+^2$. Then, for all $(\Delta T_1, \Delta T_2) \in R_+^2$ such that $\Delta T_1 \neq \Delta T_2$, we get

$$\frac{\Delta T_2 - \Delta T_1}{\ln \dfrac{\Delta T_2}{\Delta T_1}} = \frac{\Delta T_2 - \Delta T_1}{2 \displaystyle\sum_{i=0}^{\infty} \frac{1}{2i+1} \left(\frac{\Delta T_2 - \Delta T_1}{\Delta T_2 + \Delta T_1} \right)^{2i+1}}$$

$$= \frac{\Delta T_2 - \Delta T_1}{2 \left(\dfrac{\Delta T_2 - \Delta T_1}{\Delta T_2 + \Delta T_1} \right) \left[1 + \displaystyle\sum_{i=1}^{\infty} \frac{1}{2i+1} \left(\frac{\Delta T_2 - \Delta T_1}{\Delta T_2 + \Delta T_1} \right)^{2i} \right]}$$

$$= \frac{\dfrac{\Delta T_1 + \Delta T_2}{2}}{1 + \displaystyle\sum_{i=1}^{\infty} \frac{1}{2i+1} \left(\frac{\Delta T_2 - \Delta T_1}{\Delta T_2 + \Delta T_1} \right)^{2i}}$$

$$= \frac{\Delta T_a(\Delta T_1, \Delta T_2)}{L(\Delta T_1, \Delta T_2)} \qquad \text{(from (2) and (29))}$$

2) $\Delta T_1 = \Delta T_2$. Notice from (29) and (2) that $L(\Delta T, \Delta T) = 1$ and $\Delta T_a(\Delta T, \Delta T) = \Delta T$, $\forall \Delta T \neq 0$. Then, for $\Delta T_1 = \Delta T_2 = \Delta T > 0$, we have $\dfrac{\Delta T_a(\Delta T, \Delta T)}{L(\Delta T, \Delta T)} = \Delta T = \Delta T_L(\Delta T, \Delta T)$.

Remark 1. Lemma 1 is of great help in the analysis of ΔT_L. Indeed, since $L(\Delta T, \Delta T) \geq 1$

[3] Formula 4.1.27 in (Abramowitz and Stegun, 1972) states the following well-known (infinite) series expansion of the logarithmic function: $\ln z = 2 \sum_{i=0}^{\infty} \frac{1}{2i+1} \left(\frac{z-1}{z+1} \right)^{2i+1}, \forall z : \Re z \geq 0, z \neq 0.$

for all $(\Delta T_1, \Delta T_2)$ such that $\Delta T_1 + \Delta T_2 \neq 0$ (see (29)), one sees readily that $\frac{\Delta T_a(\Delta T_1, \Delta T_2)}{L(\Delta T_1, \Delta T_2)}$ is well-defined at every point on R_+^2. Actually, it is easy to see that $(\Delta T_1, \Delta T_2) \to (\Delta T, \Delta T) \Rightarrow L(\Delta T_1, \Delta T_2) \to L(\Delta T, \Delta T) = 1$, $\forall \Delta T \neq 0$. Consequently, one sees from (30) that $\Delta T_1 - \Delta T_2 \to 0 \Rightarrow \Delta T_L(\Delta T_1, \Delta T_2) \to \Delta T_a(\Delta T_1, \Delta T_2)$ explaining why such an assertion is suggested by several authors, like in (Kern, 1950; Gardner and Taborek, 1977; Faires, 1957). Moreover, from (30), we have

$$\frac{\partial \Delta T_L}{\partial \Delta T_i} = \frac{\frac{L}{2} - \Delta T_a \frac{\partial L}{\partial \Delta T_i}}{L^2} \tag{31}$$

i=1,2, where the arguments have been dropped for the sake of simplicity, and from (29), one sees that $\frac{\partial \Delta T_L}{\partial \Delta T_i} = \sum_{i=1}^{\infty} \frac{2i}{2i+1} S^{2i-1} \frac{\partial S}{\partial \Delta T_i}$, with $S = \frac{\Delta T_2 - \Delta T_1}{\Delta T_2 + \Delta T_1}$ and $\frac{\partial S}{\partial \Delta T_i} = \frac{(-1)^i 2\Delta T_{3-i}}{(\Delta T_2 + \Delta T_1)^2}$. Hence,

$$\Delta T_1 - \Delta T_2 \to 0 \Rightarrow \left(\frac{\partial L}{\partial \Delta T_i}, L \right) \to (0,1) \Rightarrow \frac{\partial \Delta T_L}{\partial \Delta T_i} \to \frac{1}{2},$$

i=1,2, as claimed by authors like in [16]. Furthermore, synthetically dividing $\Delta T_a(\Delta T_1, \Delta T_2)$ by $L(\Delta T_1, \Delta T_2)$, one gets

$$\frac{\Delta T_a(\Delta T_1, \Delta T_2)}{L(\Delta T_1, \Delta T_2)} = \Delta T_a(\Delta T_1, \Delta T_2) - \frac{1}{6} \frac{(\Delta T_1 - \Delta T_2)^2}{\Delta T_1 + \Delta T_2} - \frac{2}{45} \frac{(\Delta T_1 - \Delta T_2)^4}{(\Delta T_1 + \Delta T_2)^3} - \ldots \tag{32}$$

coinciding with the power series expansion suggested in (Steiner, 1989) as a replacement for the LMTD. The proofs of some other analytical features of ΔT_L stated in the present section are simplified through the use of (30).

Remark 2. Expression (32) is helpful to see the link of the replacement formulas (5)–(8) with the LMTD. Indeed, notice that for close enough values of ΔT_1 and ΔT_2, the LMTD may be approximated neglecting the third and upper terms of the series, *i.e.* $\frac{\Delta T_a(\Delta T_1, \Delta T_2)}{L(\Delta T_1, \Delta T_2)} \approx \Delta T_a(\Delta T_1, \Delta T_2) - \frac{1}{6} \frac{(\Delta T_1 - \Delta T_2)^2}{\Delta T_1 + \Delta T_2}$. For instance, ΔT_1 and ΔT_2 resulting in a sufficiently small value of $(\sqrt{\Delta T_1} - \sqrt{\Delta T_2})^2$, say $(\sqrt{\Delta T_1} - \sqrt{\Delta T_2})^2 \approx 0$ (observe that $(\sqrt{\Delta T_1} - \sqrt{\Delta T_2})^2 \approx 0 \Leftrightarrow \Delta T_1 \approx \Delta T_2$), may be considered to apply for such an approximation. Thus:

$$(\sqrt{\Delta T_1} - \sqrt{\Delta T_2})^2 \approx 0 \Leftrightarrow 2\sqrt{\Delta T_1 \Delta T_2} \approx \Delta T_1 + \Delta T_2 \Leftrightarrow$$
$$\Delta T_1 + 2\sqrt{\Delta T_1 \Delta T_2} + \Delta T_2 \approx 2(\Delta T_1 + \Delta T_2) \Leftrightarrow$$
$$(\sqrt{\Delta T_1} + \sqrt{\Delta T_2})^2 \approx 2(\Delta T_1 + \Delta T_2) \Leftrightarrow \frac{(\sqrt{\Delta T_1} + \sqrt{\Delta T_2})^2}{2(\Delta T_1 + \Delta T_2)}$$
$$\approx 1 \Leftrightarrow \frac{(\sqrt{\Delta T_1} + \sqrt{\Delta T_2})^2}{3} \frac{(\sqrt{\Delta T_1} + \sqrt{\Delta T_2})^2}{2(\Delta T_1 + \Delta T_2)}$$
$$- \approx - \frac{(\sqrt{\Delta T_1} + \sqrt{\Delta T_2})^2}{3} \Leftrightarrow -\frac{1}{6} \frac{(\Delta T_1 + \Delta T_2)^2}{\Delta T_1 + \Delta T_2} \approx$$
$$-\frac{2}{3}\left(\frac{\Delta T_1 + \Delta T_2}{2} - \sqrt{\Delta T_1 \Delta T_2} \right) \Leftrightarrow \frac{\Delta T_1 + \Delta T_2}{2} -$$
$$\frac{1}{6} \frac{(\Delta T_1 + \Delta T_2)^2}{\Delta T_1 + \Delta T_2} \approx \frac{\Delta T_a + 2\Delta T_g}{3},$$

i.e. $\frac{\Delta T_a(\Delta T_1, \Delta T_2)}{L(\Delta T_1, \Delta T_2)} \approx \Delta T_{\ell P}$, (see (5)). Alternatively, for ΔT_1 and ΔT_2 resulting in $a_1 = \left(5\Delta T_1^{2/3} + 11\Delta T_1^{1/3}\Delta T_2^{1/2} + 5\Delta T_2^{2/3}\right) \times \left(\Delta T_1^{1/3} - \Delta T_2^{1/3}\right)^2 \approx 0$ (observe that $a_1 \approx 0 \Rightarrow \left(\Delta T_1^{1/3} - \Delta T_2^{1/3}\right)^2 \approx 0 \Leftrightarrow \Delta T_1 \approx \Delta T_2$), we have

$$a_1 \approx 0 \Leftrightarrow \frac{\Delta T_1 + \Delta T_2}{2} - \frac{1}{6} \frac{\left(\Delta T_1 - \Delta T_2\right)^2}{\Delta T_1 + \Delta T_2} \approx \left(\frac{\Delta T_1^{1/3} + \Delta T_2^{1/3}}{2} \right)^3$$

(see Appendix A), *i.e.* $\frac{\Delta T_a(\Delta T_1, \Delta T_2)}{L(\Delta T_1, \Delta T_2)} \approx \Delta T_{\ell U}$ (see (6)). Furthermore, for ΔT_1 and ΔT_2 such that $a_2 = \left[2\left(\frac{\Delta T_1 + \Delta T_2}{2} \right)^{2/3} + \left(\Delta T_1 \Delta T_2\right)^{1/3} \right] \cdot \left[2\left(\frac{\Delta T_1 + \Delta T_2}{2} \right)^{2/3} - 2(\Delta T_1 \Delta T_2)^{1/3} \right] \approx 0$ (observe that

$$a_2 \approx 0 \Rightarrow \left[2\left(\frac{\Delta T_1 + \Delta T_2}{2} \right)^{2/3} - 2(\Delta T_1 \Delta T_2)^{1/3} \right] \approx 0 \Leftrightarrow$$

$\left(\frac{\Delta T_1 + \Delta T_2}{2}\right)^{2/3} \approx (\Delta T_1 \Delta T_2)^{1/3} \Leftrightarrow \left(\frac{\Delta T_1 + \Delta T_2}{2}\right)^2 \approx \Delta T_1 \Delta T_2 \Leftrightarrow \Delta T_1^2 + 2\Delta T_1 \Delta T_2 + \Delta T_2^2 \approx 4\Delta T_1 \Delta T_2 \Leftrightarrow (\Delta T_1 - \Delta T_2)^2 \approx 0 \Leftrightarrow \Delta T_1 \approx \Delta T_2$), we have $a_2 \approx 0 \Leftrightarrow \frac{\Delta T_1 + \Delta T_2}{2} - \frac{1}{6}\frac{(\Delta T_1 - \Delta T_2)^2}{\Delta T_1 + \Delta T_2} \approx \Delta T_a^{1/3} \Delta T_g^{2/3}$ (see Appendix B), *i.e.* $\frac{\Delta T_a(\Delta T_1, \Delta T_2)}{L(\Delta T_1, \Delta T_2)} \approx \Delta T_{\ell C1}$ (see (7)). From these developments we see that as long as close enough values of ΔT_1 and ΔT_2 are considered, relatively good approximations of the LMTD are gotten through the replacement expressions (5)–(8) (recall that (8) is just a refinement of (6)). On the contrary, such approximations deteriorate as ΔT_1 and ΔT_2 are taken far from each other.

Remark 3. Let us point out that the quotient function $\frac{\Delta T_a}{L}(\Delta T_1, \Delta T_2) = \frac{\Delta T_a(\Delta T_1, \Delta T_2)}{L(\Delta T_1, \Delta T_2)}$ is defined on a subset wider than the domain of ΔT_L. Actually, $\frac{\Delta T_a}{L} : \{(\Delta T_1, \Delta T_2) \in R^2 | \Delta T_1 + \Delta T_2 \neq 0\} \to R$. Then $\frac{\Delta T_a}{L}$ is an extension of ΔT_L (actually Lemma 1 can be synthesized as $\Delta T_L \equiv \left.\frac{\Delta T_a}{L}\right|_{R_+^2}$). Therefore, $\frac{\Delta T_a}{L}$ may be used to extrapolate ΔT_L to points on R^2 where the latter is not defined. This may not make physical sense but could be helpful for analysis purposes. For example, one sees from (29) that $L\big|_{\partial R_+^2} = \sum_{j=0}^{\infty} \frac{1}{2j+1} = \sum_{i=1}^{\infty} \frac{1}{2i-1}$, and since $\frac{1}{2i-1} > \frac{1}{2i}, \forall i \geq 1$, then $\sum_{i=1}^{\infty} \frac{1}{2i-1}$ is divergent according to theorems 3.28 and 3.25 in[4] (Rudin, 1976). Therefore $\frac{\Delta T_a(\Delta T_1, \Delta T_2)}{L(\Delta T_1, \Delta T_2)} \to 0$ as $(\Delta T_1, \Delta T_2) \to \partial R_+^2$ which, from Lemma 1, implies that $\Delta T_L(\Delta T_1, \Delta T_2) \to 0$ as $(\Delta T_1, \Delta T_2)$ approaches ∂R_+^2 (from the interior of R_+^2). Then, zero can be considered the value that the LMTD (as a bivariable function) would take at any point on ∂R_+^2. This has been useful in the analyses developed in (Zavala-Río *et al.*, 2003; Zavala-Río, 2004) to prove that the solutions of system (4) remain within a physically coherent (according to the assumptions made therein) domain where the outlet temperatures cannot become either higher than T_{hi} or lower than T_{ci} (see Fig. 1).

Lemma 2. *The LMTD model in (28) is continuously differentiable and positive on* R_+^2.

Proof. Since $L(\Delta T_1, \Delta T_2) \geq 1, \forall (\Delta T_1, \Delta T_2) \in R_+^2$ (see (29)), one sees from (30) (and (2)) that ΔT_L exists and is continuous on R_+^2 (also verifiable from (32), according to Lemma 1). Moreover, from (31), one observes that $\frac{\partial \Delta T_L}{\partial \Delta T_i}$, i = 1,2, exist and are continuous on R_+^2, proving continuous differentiability. On the other hand, notice (from (2)) that $\Delta T_a(\Delta T_1, \Delta T_2) > 0, \forall (\Delta T_1, \Delta T_2) \in R_+^2$ (the average of two positive numbers is positive). Consequently,

$$0 < \frac{\Delta T_a(\Delta T_1, \Delta T_2)}{L(\Delta T_1, \Delta T_2)} \leq \Delta T_a(\Delta T_1, \Delta T_2), \forall (\Delta T_1, \Delta T_2) \in R_+^2 .$$

Then, from Lemma 1, positivity of ΔT_L follows too.

Lemma 2 is specially interesting when $\Delta T_1 = \Delta T_2$. It proves to be essential when the LMTD is involved in dynamical analysis and control synthesis of heat exchangers. In (Zavala-Río and Santiesteban-Cos, 2004), for example, it is shown that thanks to such continuous differentiability property, existence and

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[4] In (Rudin, 1976), Theorem 3.28 states that $\Sigma_{n=0}^{\infty} 1/n^p$ converges if p > 1 and diverges if $p \leq 1$, while point (b) of Theorem 3.25 states that if $a_n \geq d_n \geq 0$ for $n \geq N_0$ (for some N_0), and if $\Sigma_{n=0}^{\infty} d_n$ diverges, then $\Sigma_{n=0}^{\infty} a_n$ diverges.

uniqueness of solutions of heat exchanger lumped-parameter dynamical models as in (4), using the well-posed LMTD as driving force, are guaranteed. Furthermore, as mentioned in Section 1, the partial derivatives of the LMTD (with respect to its arguments) are used in the control laws synthesized in (Alsop and Edgar, 1989; Malleswararao and Chidambaram, 1992). Provided that the well-posed LMTD in (28) is used, their closed-loop analyses are correct (according to the respective assumptions made in each of those works). On the contrary, such control strategies could cause inconvenience if ΔT_L were not continuously differentiable on R_+^2, or ΔT_ℓ in (1) were used instead. This is true for any kind of (numerical) algorithm using the partial derivatives of the LMTD, like the equation-oriented flowsheeting program application mentioned by Paterson (1984) (see Section 1 above).

Lemma 3. *The LMTD model in (28) is strictly increasing in its arguments, i e.* $\frac{\partial \Delta T_L}{\partial \Delta T_i} > 0$, i=1,2, $\forall(\Delta T_1, \Delta T_2) \in R_+^2$.

Proof. From (28) (for $\Delta T_2 \neq \Delta T_1$) and (31) (for $\Delta T_2 = \Delta T_1$; recall Remark 1), we have

$$\frac{\partial \Delta T_L}{\partial \Delta T_i} = \begin{cases} \dfrac{(-1)^i \left[ln\dfrac{\Delta T_2}{\Delta T_1} - \dfrac{\Delta T_2 - \Delta T_1}{\Delta T_i} \right]}{\left[ln\dfrac{\Delta T_2}{\Delta T_1} \right]^2} & \text{if } \Delta T_1 \neq \Delta T_2 \\ \dfrac{1}{2} & \text{if } \Delta T_2 = \Delta T_1 \end{cases} \tag{33}$$

i=1,2, existing and being continuous on R_+^2 according to Lemma 2. Notice from (33) that the proof of the lemma amounts to demonstrate positivity of $(-1)^i \left[ln\frac{\Delta T_2}{\Delta T_1} - \frac{\Delta T_2 - \Delta T_1}{\Delta T_i} \right]$, i=1,2, for all $\forall(\Delta T_1, \Delta T_2) \in R_+^2$ such that $\Delta T_1 \neq \Delta T_2$. Then, from Formula 4.1.33 in [23], we have, for all such $(\Delta T_1, \Delta T_2)$ that $\frac{\Delta T_2 - \Delta T_1}{\Delta T_2} < \ln\frac{\Delta T_2}{\Delta T_1} < \frac{\Delta T_2 - \Delta T_1}{\Delta T_1} \Leftrightarrow$ $(-1)^i \left[\ln\frac{\Delta T_2}{\Delta T_1} - \frac{\Delta T_2 - \Delta T_1}{\Delta T_i} \right] > 0$, i=1,2, proving the lemma.

Lemma 3 has been helpful in (Zavala-Río and Santiesteban-Cos, 2004) to prove the existence of a unique equilibrium solution of system (4) to which every trajectory converges. Furthermore, it has been helpful in (Zavala-Río *et al.*, 2003) to characterize heat exchangers as cooperative systems (under the assumptions made therein). This means that a temperature raise at any of the outlets entails a temperature increase at the other outlet. The dynamics of heat exchangers may then be analyzed under the framework of cooperative systems, which constitutes a complementary way to comprehend their behaviour.

Discussion and conclusions

Some insights on the usefulness of the results presented above are in order. For instance, a well-known calculation problem is that of finding a suitable ΔT_2 satisfying an LMTD relation to a given ΔT_1 (Paterson, 1984). An exact solution to such a problem may be gotten considering the complete LMTD expression in (28). Indeed, departing from (16)-(19), (28), one gets $\Delta T_2 = e^{UA\left(\frac{\alpha}{F_c C_{pc}} - \frac{1}{F_h C_{ph}}\right)} \Delta T_1$, which is a simple linear expression. Numerical algorithms (whose convergence is not even guaranteed) or nonlinear complex approximation expressions (Paterson, 1984; Chen, 1987), are not any longer needed to perform such a calculation. Furthermore, the results developed in Sections 3 and 4 have been very helpful to characterize dynamic properties of heat exchangers through simple lumped-parameter dynamic models (Zavala-Río and Santiesteban-Cos, 2004; Zavala-Río *et al.*, 2003). In (Zavala-Río and Santiesteban-Cos, 2004), for instance, dynamic properties of model (4), using (28), have been analyzed and proved to be *equivalent* to those of the distributed-parameter model where (4) comes from. The complete expression of (28) and its analytical features have played a key role for such results in three main directions: 1) existence and uniqueness of solutions of (4) have been demonstrated for any physically possible initial state conditions (such is not a property of (4) if (1) is used); 2) (4) was proved to have a unique equilibrium point, $T_o^* = (T_{co}^*, T_{ho}^*)^T$, linearly related to the inlet temperatures, $T_i = (T_{ci}, T_{hi})^T$, *i.e.* $T_o^* = AT_i$, where the complete expressions of the elements of $A \in R^{2\times 2}$ have been obtained in terms of the system properties, whatever value these ones take; 3) such a unique equilibrium point was proved to be exponentially stable, and to keep its asymptotical stability character globally on the system state-space domain. On the other hand, based on the results in Section 3, other interesting dynamic properties of system (4) have been proved in (Zavala-Río *et al.*, 2003). Basically, under certain standard assumptions, model (4) has been characterized as a positive, compartmental, cooperative system. These features provide a new framework for the comprehension of the dynamic behavior of heat exchangers. Furthermore, the use of the LMTD has found an important application in dynamic simulation too (Papastratos *et al.*; Zeghal *et al.*, 1991; Reimann, 1986). However, none of the previous works on the subject treat the inconsistency of (1). Results avoiding such an inconsistency are generally shown. The complete expression in (28) may now be used avoiding such a problem. Other numerical problems arising from the choice of the system parameter values may still take place if simulations are performed through a simple model as (4), like numerical rigidity, for instance. Nevertheless, such problems may be avoided if more than one *bi-compartmental cell* is considered in the lumped-parameter modelling, as suggested in (Papastratos, *et al.*; Zeghal *et al.*, 1991; Reimann, 1986). Finally, the results proposed in this work find potential applications in control design of heat exchangers too. For instance, based on (4), taking F_c as input and T_{ho} as output, input-output (partial) linearization control algorithms have been proposed in (Alsop and Edgar, 1989; Malleswararao and Chidambaram, 1992). In a natural way, such design methods lead the proposed control laws to involve the partial derivatives of ΔT_1 and ΔT_2. The authors use (1) but do not treat its inconsistency either. Again, numerical results avoiding such inconsistencies are presented. The results developed here give sense to their results if (28), and (33), are considered in their analysis and their proposed algorithms.

Thus, an analytical study of the LMTD turns out to be essential in view of the important role that the LMTD plays in several theoretical and practical aspects of heat exchangers. The results developed here are intended to support all those works where the LMTD is involved and the consideration of its analytical properties is important, like dynamical modelling, simulation, characterization, and control of heat exchangers.

A. Developments for Underwood's approximation

$$\left(5\Delta T_1^{2/3} + 11\Delta T_1^{1/3}\Delta T_2^{1/3} + 5\Delta T_2^{2/3}\right)\left(\Delta T_1^{1/3} - \Delta T_2^{1/3}\right)^2 \approx 0 \Leftrightarrow [5(\Delta T_1^{2/3} + \Delta T_1^{1/3}\Delta T_2^{1/3} + 5\Delta T_2^{2/3}) + 6\Delta T_1^{1/3}\Delta T_2^{1/3}]$$

$$(\Delta T_1^{1/3} - \Delta T_2^{1/3})^2 \approx 0 \Leftrightarrow [5(\Delta T_1 - \Delta T_2) + 6(\Delta T_1^{2/3}\Delta T_2^{1/3} - \Delta T_1^{1/3}\Delta T_2^{2/3})](\Delta T_1^{1/3} - \Delta T_2^{1/3}) \approx 0 \Leftrightarrow 5\Delta T_1^{4/3} + \Delta T_1 \Delta T_2^{1/3} - 12\Delta T_1^{2/3}\Delta T_2^{2/3} + \Delta T_1^{1/3}\Delta T_2 + 5\Delta T_2^{4/3} \approx 0 \Leftrightarrow 4\Delta T_1^{4/3} + 4\Delta T_1^{2/3}\Delta T_2^{2/3} + 4\Delta T_2^{4/3} + 8\Delta T_1 \Delta T_2^{1/3} + 8\Delta T_1^{2/3}\Delta T_2^{2/3}$$

$$+8\Delta T_1^{1/3}\Delta T_2 \approx 9\Delta T_1^{4/3} + 9\Delta T_1\Delta T_2^{1/3} + 9\Delta T_1^{1/3}\Delta T_2 + 9\Delta T_2^{4/3}$$
$$\Leftrightarrow 4(\Delta T_1^{2/3} + \Delta T_1^{1/3}\Delta T_2^{1/3} + \Delta T_2^{2/3})^2 \approx 9(\Delta T_1 + \Delta T_2)(\Delta T_1^{1/3}$$
$$+\Delta T_2^{1/3}) \Leftrightarrow \frac{4(\Delta T_1^{2/3} + \Delta T_1^{1/3}\Delta T_2^{1/3} + \Delta T_2^{2/3})^2}{9(\Delta T_1 + \Delta T_2)} \approx \Delta T_1^{1/3} + \Delta T_2^{1/3}$$
$$\Leftrightarrow -\frac{3}{8}\left(\Delta T_1^{1/3} + \Delta T_2^{1/3}\right)^2 \cdot \frac{4\left(\Delta T_1^{2/3} + \Delta T_1^{1/3}\Delta T_2^{1/3} + \Delta T_2^{2/3}\right)^2}{9(\Delta T_1 + \Delta T_2)}$$
$$\approx -\frac{3}{8}(\Delta T_1^{1/3} - \Delta T_2^{1/3})^2(\Delta T_1^{1/3} + \Delta T_2^{1/3}) \Leftrightarrow -\frac{1}{6}\frac{(\Delta T_1 - \Delta T_2)^2}{\Delta T_1 + \Delta T_2}$$
$$\approx \frac{\Delta T_1^{1/3} + \Delta T_2^{1/3}}{2}\left[\frac{1}{4}(-3\Delta T_1^{2/3} + 6\Delta T_1^{1/3}\Delta T_2^{1/3} - 3\Delta T_2^{2/3})\right] = \frac{\Delta T_1^{1/3} + \Delta T_2^{1/3}}{2}\left[\frac{1}{4}(\Delta T_1^{2/3} + 2\Delta T_1^{1/3}\Delta T_2^{1/3} + \Delta T_2^{2/3})\right.$$
$$\left. - \Delta T_1^{2/3} + \Delta T_1^{1/3}\Delta T_2^{1/3} - \Delta T_2^{2/3}\right] = \frac{\Delta T_1^{1/3} + \Delta T_2^{1/3}}{2} \times$$
$$\left[\left(\frac{\Delta T_1^{1/3} + \Delta T_2^{1/3}}{2}\right)^2 - \left(\Delta T_1^{2/3} - \Delta T_1^{1/3}\Delta T_2^{1/3} + \Delta T_2^{2/3}\right)\right] = \left(\frac{\Delta T_1^{1/3} + \Delta T_2^{1/3}}{2}\right)^3 - \frac{\Delta T_1 + \Delta T_2}{2} \Leftrightarrow$$
$$\frac{\Delta T_1 + \Delta T_2}{2} - \frac{1}{6}\frac{\left(\Delta T_1 - \Delta T_2\right)^2}{\Delta T_1 + \Delta T_2} \approx \left(\frac{\Delta T_1^{1/3} + \Delta T_2^{1/3}}{2}\right)^3.$$

B. Developments for Chen's approximation

$$\left[2\left(\frac{\Delta T_1 + \Delta T_2}{2}\right)^{2/3} + (\Delta T_1\Delta T_2)^{1/3}\right]\left[2\left(\frac{\Delta T_1 + \Delta T_2}{2}\right)^{2/3} - 2(\Delta T_1\Delta T_2)^{1/3}\right] \approx 0 \Leftrightarrow \left[2\left(\frac{\Delta T_1 + \Delta T_2}{2}\right)^{2/3} -\right.$$
$$\left. -\frac{(\Delta T_1\Delta T_2)^{1/3}}{2}\right] - \frac{9}{4}(\Delta T_1\Delta T_2)^{2/3} \approx 0 \Leftrightarrow 4\left(\frac{\Delta T_1 + \Delta T_2}{2}\right)^{4/3} - 2(\Delta T_1\Delta T_2)^{1/3}\frac{(\Delta T_1 + \Delta T_2)^{2/3}}{2} - 2(\Delta T_1\Delta T_2)^{2/3}$$
$$\approx 0 \Leftrightarrow 2\left[\left(\frac{\Delta T_1 + \Delta T_2}{2}\right)^{4/3} + (\Delta T_1\Delta T_2)^{1/3}\frac{(\Delta T_1 + \Delta T_2)^{2/3}}{2} + (\Delta T_1\Delta T_2)^{2/3}\right] \approx 6\left(\frac{\Delta T_1 + \Delta T_2}{2}\right)^{4/3} =$$
$$3(\Delta T_1 + \Delta T_2)\left(\frac{\Delta T_1 + \Delta T_2}{2}\right)^{1/3} \Leftrightarrow -\frac{1}{6}\frac{\left(\Delta T_1 + \Delta T_2\right)^2}{\Delta T_1 + \Delta T_2} \times \left[\left(\frac{\Delta T_1 + \Delta T_2}{2}\right)^{4/3} + (\Delta T_1\Delta T_2)^{1/3}\frac{(\Delta T_1 + \Delta T_2)^{2/3}}{2}\right.$$
$$\left. + (\Delta T_1\Delta T_2)^{2/3}\right] \approx -\left(\frac{\Delta T_1 - \Delta T_2}{2}\right)^2\left(\frac{\Delta T_1 + \Delta T_2}{2}\right)^{1/3} = \left[\Delta T_1\Delta T_2 - \left(\frac{\Delta T_1 + \Delta T_2}{2}\right)^2\right]\left(\frac{\Delta T_1 + \Delta T_2}{2}\right)^{1/3}$$
$$\Leftrightarrow -\frac{1}{6}\frac{\left(\Delta T_1 - \Delta T_2\right)^2}{\Delta T_1 + \Delta T_2} \approx \left[\left(\Delta T_1\Delta T_2\right)^{1/3} - \left(\frac{\Delta T_1 + \Delta T_2}{2}\right)^{2/3}\right] \times \left(\frac{\Delta T_1 + \Delta T_2}{2}\right)^{1/3} = \left(\frac{\Delta T_1 + \Delta T_2}{2}\right)^{1/3}\left(\Delta T_1\Delta T_2\right)^{1/3} -$$
$$\frac{\Delta T_1 + \Delta T_2}{2} \Leftrightarrow \frac{\Delta T_1 + \Delta T_2}{2} - \frac{1}{6}\frac{\left(\Delta T_1 - \Delta T_2\right)^2}{\Delta T_1 + \Delta T_2} \approx \left(\frac{\Delta T_1 + \Delta T_2}{2}\right)^{1/3}\left(\Delta T_1\Delta T_2\right)^{1/3}.$$

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