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Numerical Computation of Lyapunov Matrices for Integral Delay Systems

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Abstract: This paper focuses on the problem of computing Lyapunov matrices for integral delay systems. It is shown that these Lyapunov matrices cannot be computed by means of the existing methods for Lyapunov matrices of differential delay systems. Then, a numerical algorithm for constructing piecewise linear approximations of Lyapunov matrices for integral delay systems is proposed. An example illustrates the procedure. *Copyright ©2017 IFAC*

Keywords: Integral delay systems, Lyapunov-Krasovskii functionals, Lyapunov matrices.

1. INTRODUCTION

Recently in Melchor-Aguilar et al. (2010), direct and converse Lyapunov-Krasovskii theorems for the exponential stability of integral delay systems have been introduced. It has been shown there that a new type of Lyapunov functionals is required in order to properly address the dynamics of such class of systems. General expressions of quadratic Lyapunov-Krasovskii functionals with a given time derivative were provided. These functionals are defined by special matrix-valued functions which are the counterpart of the Lyapunov matrices that appear in the computation of Lyapunov-Krasovskii functionals of the complete type for differential delay systems Kharitonov and Zhabko (2003); therefore, it appears natural to call such matrix-valued functions as *Lyapunov matrices for integral delay systems* and the corresponding functionals as *Lyapunov-Krasovski functionals of complete type for integral delay systems.*

However, in contrast with the case of differential delay systems for which several semi-analytical and/or numerical methods of computing Lyapunov matrices exist, see the recent book Kharitonov (2013) for a complete description of such methods, to the best of our knowledge, no computational procedures for constructing the Lyapunov matrices of integral delay systems has been proposed in the literature.

The lack of numerical algorithms for computing Lyapunov matrices of integral delay systems has limited the application of the complete type functionals but, at the same time, it has motivated the construction of reduced type functionals to obtain stability and robust stability conditions formulated directly in terms of the coefficients of integral delay systems expressed as linear matrix inequalities, see for instance Melchor-Aguilar (2010), Mondie and Melchor-Aguilar (2012) and Melchor-Aguilar (2014).

In this paper, we address the problem of computing Lyapunov matrices for integral delay systems. Firstly, we show that the existing methods for differential delay systems do not provide a solution to such a problem. Then, we present a numerical scheme for computing piecewise linear approximations of Lyapunov matrices of integral delay systems with a constant matrix kernel and one delay. We also provide a method to measure the quality of the computed approximations.

The remaining part of the paper is organized as follows. Section 2 presents some preliminaries. The Lyapunov functionals and matrices for integral delay systems are introduced. Section 3 is devoted to show that the numerical methods for delay differential systems do not allow us to compute Lyapunov matrices for integral delay systems. The numerical algorithms for computing piecewise linear approximations of Lyapunov matrices and the approximation error are respectively given in sections 4 and 5. An example illustrating the algorithm is provided in section 6 and some concluding remarks end the paper.

2. PRELIMINARIES

Consider the integral delay system

$$x(t) = F \int_{-h}^{0} x(t+\theta)d\theta, \tag{1}$$

where $F \in \mathbb{R}^n$ and $h > 0$. Given any initial function $\varphi \in \mathcal{PC} = \mathcal{PC}\left([-h,0), \mathbb{R}^n\right)$, the space of piecewise continuous bounded functions mapping the interval $[-h,0)$ to $\mathbb{R}^n$, there exists a unique solution $x(t,\varphi)$ of (1) which is defined for all $t \in [-h,\infty)$, see Melchor-Aguilar et al. (2010).

Definition 1. Melchor-Aguilar et al. (2010) System (1) is said to be exponentially stable if there exist $\alpha > 0$ and $\mu > 0$ such that every solution of (1) satisfies the inequality

$$\|x(t,\varphi)\| \leq \mu e^{-\alpha t} \|\varphi\|_h, \quad \forall t \geq 0,$$

where $\|\varphi\|_h = \sup_{\theta \in [-h,0)} \|\varphi(\theta)\|$.

As usual for delay systems, we define the natural state of (1) by $x_t(\theta,\varphi) \triangleq x(t+\theta,\varphi), \theta \in [-h,0)$. For simplicity of the notation, one writes $x_t(\varphi)$ instead of $x_t(\theta,\varphi), \theta \in [-h,0)$. Also, when the initial function is irrelevant from the context, we simply write $x(t)$ and x_t instead of $x(t,\varphi)$ and $x_t(\varphi)$.

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Theorem 1. Melchor-Aguilar et al. (2010) System (1) is exponentially stable if there exists a continuous functional $v : \mathcal{PC} \to \mathbb{R}$ such that $t \to v(x_t(\varphi))$ is differentiable and the following conditions hold:

(1) $\alpha_1 \int_{-h}^{0} \|\varphi(\theta)\|^2 d\theta \le v(\varphi) \le \alpha_2 \int_{-h}^{0} \|\varphi(\theta)\|^2 d\theta$, for some constants $0 < \alpha_1 \le \alpha_2$,

(2) $\frac{d}{dt} v(x_t(\varphi)) \le -\beta \int_{-h}^{0} \|x(t+\theta, \varphi)\|^2 d\theta$, for a constant $\beta > 0$.

Let $K(t)$ be the solution of the matrix equation

$$K(t) = \left(\int_{-h}^{0} K(t+\theta) d\theta \right) F,$$

with the initial condition $K(t) = -K_0, t \in [-h, 0)$, where $K_0 = (I - hF)^{-1}$. The matrix $K(t)$ is known as the fundamental matrix of the system (1), see Melchor-Aguilar et al. (2010).

Let us assume that (1) is exponentially stable. Given $W = W^T$ define the matrix

$$U(\tau) \triangleq \int_{0}^{\infty} K^T(t) W K(t+\tau) dt, \quad \tau \in [-h, h]. \tag{2}$$

Note that the exponential stability of (1) guarantees the existence of the improper integral in (2).

By assuming the exponential stability of (1) and defining on $\mathcal{PC}$ the functional

$$w(\varphi) = \varphi^T(-h) W_0 \varphi(-h) + \int_{-h}^{0} \varphi^T(\theta) W_1 \varphi(\theta) d\theta,$$

where W_0 and W_1 are any positive definite matrices, the functional

$$\begin{aligned} v(x_t) = &\left(F \int_{-h}^{0} x(t+\theta) d\theta \right)^T U(0) \left(F \int_{-h}^{0} x(t+\theta) d\theta \right) \\ &-2 \left(F \int_{-h}^{0} x(t+\theta) d\theta \right)^T \int_{-h}^{0} U(-h-\theta) F x(t+\theta) d\theta \\ &+ \int_{-h}^{0} x^T(t+\theta_1) F^T \left(\int_{-h}^{0} U(\theta_1 - \theta_2) F x(t+\theta_2) d\theta_2 \right) d\theta_1 \\ &- \int_{-h}^{0} x^T(t+\theta_1) F^T K_0^T W \times \\ &\times \left[\int_{-h}^{0} \left(\int_{-h-\theta_2}^{\theta_1 - \theta_2} K(\xi) d\xi \right) F x(t+\theta_2) d\theta_2 \right] d\theta_1 \\ &+ \int_{-h}^{0} x^T(t+\theta) \left[W_0 + (\theta + h) W_1 \right] x(t+\theta) d\theta, \end{aligned} \tag{3}$$

where $U(\cdot)$ is given by (2) with $W = W_0 + hW_1$, satisfies the equation

$$\frac{d}{dt} v(x_t) = -w(x_t), \quad t \ge 0.$$

It is shown in Melchor-Aguilar et al. (2010) that if the system (1) is exponentially stable then the functional (3) satisfies the conditions of Theorem 1.

It follows from (3) that the matrix-valued function $U(\cdot)$ is fundamental for constructing the functional $v(x_t)$. This special characteristic of the functional (3) is analogous to that of the so-called Lyapunov-Krasovskii functionals of the complete type for linear differential delay systems which construction depend on a matrix-valued function called as Lyapunov matrix for differential delay systems, see Kharitonov (2013). Thus, it appears natural to call the functional $v(x_t)$ defined by (3) as *Lyapunov-Krasovskii functional of the complete type* for the integral delay system (1) and the matrix-valued function $U(\tau)$ defined by (2) as *Lyapunov matrix* for the integral delay system (1). We formally state the Lyapunov matrix concept in the following definition.

Definition 2. The matrix (2) is a Lyapunov matrix of the system (1) associated with a symmetric matrix W.

Remark 1. The fundamental matrix $K(t)$ presents a jump discontinuity at $t = 0$ given by

$$\Delta K(0) = K(0) - K(-0) = I - K_0 - (-K_0) = I.$$

On the other hand, the Lyapunov matrix $U(\tau)$ is continuous for all $\tau \in [-h, h]$.

Lemma 2. Melchor-Aguilar et al. (2010) The Lyapunov matrix $U(\tau)$ satisfies the following conditions:

$$U(\tau) = \left(\int_{-h}^{0} U(\tau+\theta) d\theta \right) F, \qquad \tau \ge 0. \tag{4}$$

$$U(\tau) = K_0^T W \int_{0}^{\tau} K(\xi) d\xi + U^T(-\tau), \quad \tau \in [0, h]. \tag{5}$$

$$\begin{aligned} -K^T(0) W K(0) = &\left[U(0) F - U(-h) F \right]^T \\ &+ \left[U(0) F - U(-h) F \right]. \end{aligned} \tag{6}$$

We respectively call the conditions (4), (5) and (6) as the *dynamic property*, *the symmetry property* and the *algebraic property* of the Lyapunov matrix.

Clearly, these three properties provide an alternative more practical way to compute the Lyapunov matrix than the improper integral definition in (2). The dynamic property defines $U(\tau)$ as a solution of the integral delay equation (4). In order to compute such a solution one needs to know the corresponding initial condition but, however, this is not explicitly given. Thus, as occurs in the differential delay case, one of the main problems on computing Lyapunov matrices for integral delay systems consists in determining the corresponding initial condition for the integral delay equation (4).

There are some methods for determining the corresponding initial condition for Lyapunov matrices of differential delay systems Kharitonov (2013). Unfortunately, such methods cannot be directly applied to the computation of Lyapunov matrices for integral delay systems. The main problem is that in order to apply such methods one needs to differentiate the integral delay equation (4) and this leads to an unstable delay differential matrix equation, see Melchor-Aguilar et al. (2010) for details.

On the other hand, it seems natural to analyze the possibility of applying not the methods for differential delay systems but the main ideas behind them to the case of integral delay systems. Thus, in the next section, we will apply the main ideas exposed in Garcia-Lozano and Kharitonov (2006) to the numerical construction of Lyapunov matrices for integral delay systems.

3. APPLICATION OF THE NUMERICAL METHOD FOR DIFFERENTIAL DELAY SYSTEMS

Following Garcia-Lozano and Kharitonov (2006) let us define the matrix-valued function $\Phi(\tau), \tau \in [-h, 0]$, as the unknown initial function for the dynamic equation (4) and divide the interval $[-h, 0]$ into N equal segments $[-(j+1)r, -jr], j = 0, 1, 2, \ldots, N-1$, where $r = \frac{h}{N}$. Now, we introduce $N+1$ unknown matrices $\Phi_j = \Phi(-jr), j = 0, 1, 2, \ldots, N$, and define the continuous piecewise linear approximation of the initial function $\Phi(\tau)$ as follows:

$$\hat{\Phi}(s) = \left(1 + \frac{s + jr}{r}\right)\Phi_j + \left(-\frac{s + jr}{r}\right)\Phi_{j+1}, \quad (7)$$

where $s \in [-(j+1)r, -jr], j = 0, 1, 2, \ldots, N-1$.

Let $U(\tau) = U(\tau, \Phi)$ be the solution of (4) corresponding to the initial function Φ and define the continuous piecewise linear approximation of $U(\tau)$ as follows:

$$\hat{U}(s) = \left(1 - \frac{s - jr}{r}\right)U_j + \left(\frac{s - jr}{r}\right)U_{j+1}, \quad (8)$$

where $s \in [jr, (j+1)r], U_j = U(jr), j = 0, 1, 2, \cdots, N-1$. From (4), we have

$$U_j = \left(\int_{(j-N)r}^{jr} U(\xi)d\xi\right)F. \quad (9)$$

Then, according with the method in Garcia-Lozano and Kharitonov (2006), we compare U_j and U_{j+1} to obtain

$$U_{j+1} - U_j = \left(\int_0^r \left(U(jr + \theta) - \Phi((j-N)r + \theta)\right)d\theta\right)F.$$

Substituting the matrices $U(jr + \theta)$ and $\Phi((j-N)r + \theta)$ under the integral by their piecewise linear approximations (8) and (7), and using the symmetry property (5) at the partition points, i.e.,

$$U_j = \Phi_j^T + K_0^T W V_j, j = 0, 1, \ldots, N, \quad (10)$$

where

$$V_j = \int_0^{jr} K(\xi)d\xi, \quad (11)$$

one arrives at the following set of N linear equations expressed in terms of the unknown matrices $\Phi_j, j = 0, 1, \ldots, N-1$:

$$\left(\frac{r}{2}F + I\right)\Phi_j^T + \left(\frac{r}{2}F - I\right)\Phi_{j+1}^T - \frac{r}{2}\left(\Phi_{N-j} + \Phi_{N-j-1}\right)F = K_0^T W\left(V_{j+1} - V_j - \frac{r}{2}\left(V_j + V_{j+1}\right)\right). \quad (12)$$

By adding to this set the algebraic equation (6) in the partition points

$$\left(\Phi_0 - \Phi_N\right)F + F^T\left(\Phi_0 - \Phi_N\right)^T = -K^T(0)WK(0) \quad (13)$$

one arrives at the system of $N+1$ matrix equations for the $N+1$ unknown matrices $\Phi_j, j = 0, 1, \ldots, N$.

In principle, the solution of the system of equations (12)-(13) should provide the unknown matrices $\Phi_j, j = 0, 1, \ldots, N$, and the formula (7) gives the desired approximation of the initial function.

Let us consider the scalar case and two partitions of the interval $[-h, 0]$, i.e. $F \in \mathbb{R}$ and $N = 2$, that leads to $r = \frac{h}{2}$. In this case, the linear system of equations (12)-(13) can be written as $A\mathcal{X} = B$, where

$$A = \begin{bmatrix} \left(\frac{r}{2}F + 1\right) & -1 & -\frac{r}{2}F \\ -\frac{r}{2}F & 1 & \left(\frac{r}{2}F - 1\right) \\ 2F & 0 & -2F \end{bmatrix}, \quad \mathcal{X} = \begin{bmatrix}\Phi_0 & \Phi_1 & \Phi_2\end{bmatrix}^T,$$

$$B = \begin{bmatrix} K_0 W\left(V_1 - \frac{r}{2}V_1 F\right) \\ K_0 W\left((V_1 - V_1) - \frac{r}{2}\left(V_1 + V_2\right)F\right) \\ 0 \end{bmatrix}.$$

We have that $\det(A) = -2rF^2 - 2F + 2F + 2rF^2 = 0$ which implies that the matrix A is singular for all values of $F \in \mathbb{R}$ and $h > 0$. It then follows that the system of equations (12)-(13) is not consistent and, therefore, it does not provide a proper solution to the problem of computing Lyapunov matrices for integral delay systems.

4. APPROXIMATE LYAPUNOV MATRICES

The analysis in section 3 shows that a new method for calculating Lyapunov matrices of integral delay systems is required. In this section, we propose such a method.

4.1 Approximate initial function

Let us consider the equation (9). Since $(j - N)r \leq 0$, $j = 0, 1, \ldots, N$, we then can rewrite the equation (9) as

$$U_j = \left(\underbrace{\int_{(j-N)r}^{0} \Phi(\xi)d\xi}_{m_j} + \underbrace{\int_0^{jr} U(\xi)d\xi}_{n_j}\right)F. \quad (14)$$

Consider the term m_j in (14). Rewriting the integral in a sum of integrals over intervals of length r and substituting $\Phi(\xi)$ by its approximation $\hat{\Phi}(\xi)$ one gets

$$\hat{m}_j = \sum_{k=0}^{N-j-1} \int_{-(N-j-k)r}^{-(N-j-k-1)r} \hat{\Phi}(\xi)d\xi, j = 0, 1, \ldots, N-1,$$
$$\hat{m}_N = 0,$$

where $\hat{m}_j$ denotes the approximation of the term m_j. Substituting $\hat{\Phi}(\xi)$ in the integrals at the right-hand side of the expression for $\hat{m}_j$ by its piecewise linear approximation (7), one gets

$$\hat{m}_j = \tfrac{r}{2} \sum_{k=0}^{N-j-1} \left(\Phi_{N-j-k-1} + \Phi_{N-j-k}\right), j = 0, \ldots, N-1,$$
$$\hat{m}_N = 0. \quad (15)$$

Now, consider the term n_j in (14). Similarly, let us rewrite the integral as a sum of integrals over intervals of length r and substitute $U(\xi)$ by its approximation $\hat{U}(\xi)$. We have

$$\hat{n}_j = \sum_{k=0}^{j-1} \int_{(j-k-1)r}^{(j-k)r} \hat{U}(\xi)d\xi, j = 1, 2, \ldots, N,$$
$$\hat{n}_0 = 0$$

where $\hat{n}_j$ denotes the approximation of n_j. Substituting $\hat{U}(\xi)$ in the integrals at the right-hand side of the expression for $\hat{n}_j$ by its piecewise linear approximation (8), one obtains

$$\hat{n}_j = \frac{r}{2}\sum_{k=0}^{j-1}\left(U_{j-k-1}+U_{j-k}\right), \quad j=1,\ldots,N, \qquad \hat{n}_0 = 0. \tag{16}$$

From (14), we can consider the equation

$$U_j = \left(\hat{m}_j + \hat{n}_j\right)F, \quad j=0,1,\ldots,N,$$

where $\hat{m}_j$ and $\hat{n}_j$ are respectively given by (15) and (16), and accept the approximation error. By using the symmetry property at the partitions points (10) in the above equation one arrives at the following system of matrix equations:

- For $j=0$

$$\Phi_0 - \frac{r}{2}\sum_{k=0}^{N-j-1}\left(\Phi_{N-j-k-1}+\Phi_{N-j-k}\right)F = 0. \tag{17}$$

- For $j=1,2,\cdots,N-1$

$$\Phi_j^T - \frac{r}{2}\left[\sum_{k=0}^{N-j-1}\left(\Phi_{N-j-k-1}+\Phi_{N-j-k}\right)+ +\sum_{k=0}^{j-1}\left(\Phi_{j-k-1}^T+\Phi_{j-k}^T\right)\right]F = = K_0^T W\left(\frac{r}{2}\sum_{k=0}^{j-1}\left(V_{j-k-1}+V_{j-k}\right)F - V_j\right). \tag{18}$$

- For $j=N$

$$\Phi_N^T - \frac{r}{2}\sum_{k=0}^{j-1}\left(\Phi_{j-k-1}^T+\Phi_{j-k}^T\right)F = = K_0^T W\left(\frac{r}{2}\sum_{k=0}^{j-1}\left(V_{j-k-1}+V_{j-k}\right)F - V_N\right). \tag{19}$$

The equations (17)-(19) determine a system of $N+1$ matrix equations for the $N+1$ unknown matrices Φ_j, $j=0,1,\cdots,N$. The solution of this system provides matrices Φ_j, $j=0,1,2,\ldots,N$, and the formula (7) allows us to compute the desired approximation of the initial matrix function.

In order to show that the system of matrix equations (17)-(19) does not present the same inconsistent problem as the system of matrix equations (12)-(13), let us consider again the scalar case when $F\in\mathbb{R}$ and $N=2$. In this case, the system of equations (17)-(19) can be written as $\bar{A}\mathcal{X}=\bar{B}$, where

$$\bar{A} = \begin{bmatrix} \left(1-\frac{r}{2}F\right) & -rF & -\frac{r}{2}F \\ -rF & (1-rF) & 0 \\ -\frac{r}{2}F & -rF & \left(1-\frac{r}{2}F\right) \end{bmatrix}, \quad \mathcal{X} = \begin{bmatrix}\Phi_0 & \Phi_1 & \Phi_2\end{bmatrix}^T,$$

$$\bar{B} = \begin{bmatrix} 0 \\ K_0 W\left(\frac{r}{2}V_1F - V_1\right) \\ K_0 W\left(\frac{r}{2}\left(2V_1+V_2\right)F\right) - V_2 \end{bmatrix}.$$

We have that $\det(\bar{A}) = -2Fr+1$. Then, it follows that the only case when $\bar{A}$ is singular is $F=\frac{1}{2r}=\frac{1}{h}$, which is precisely the boundary of the stability region for the scalar integral delay system, see Kharitonov and Melchor-Aguilar (2000). Thus, in this case, the system of equations (17)-(19) provide us with a solution for all exponentially stable scalar integral delay systems.

Remark 2. The system of matrix equations (17)-(19) is well-defined without involving the algebraic property (6) in contrast to the method proposed in Garcia-Lozano and Kharitonov (2006) which requires the corresponding algebraic property for Lyapunov matrices of differential delay systems.

4.2 Vector form

In order to find a solution of the system of matrix equations (17)-(19) it is convenient to write it in a vector form by means of the vector operation $vec(Q)=q$, where $q\in\mathbb{R}^{n^2}$ is obtained from $Q\in\mathbb{R}^{n\times n}$ by stacking up its column. The vectorization of $C=AXB$ is $vec(C) = (A\otimes B)\,vec(X)$, where

$$A\otimes B = \begin{pmatrix} b_{11}A & b_{21}A & \cdots & b_{n1}A \\ b_{12}A & b_{22}A & \cdots & b_{n2}A \\ \vdots & \vdots & \ddots & \vdots \\ b_{1n}A & b_{2n}A & \cdots & b_{nn}A \end{pmatrix}$$

is the Kronecker product of matrix A and B, and the vectorization of $D=AX^TB$ is $vec(D) = (A\circ B)\,vec(X)$, where $A\circ B$ is defined by

$$A\circ B = \begin{pmatrix} A_1B_1^T & A_2B_1^T & \cdots & A_nB_1^T \\ A_1B_2^T & A_2B_2^T & \cdots & A_nB_2^T \\ \vdots & \vdots & \ddots & \vdots \\ A_1B_n^T & A_2B_n^T & \cdots & A_nB_n^T \end{pmatrix},$$

with $A_j, B_j, j=1,2,\ldots,N$, denoting the vector columns of A and B, respectively.

Then, the system of matrix equations (17)-(19) can be written in the vector form as follows:

- For $j=0$

$$(I\circ I)\,\phi_0 - \frac{r}{2}\sum_{k=0}^{N-j-1}(I\times F)\left(\phi_{N-j-k-1}+ +\phi_{N-j-k}\right) = 0.$$

- For $j=1,2,\cdots,N-1$

$$(I\circ F)\,\phi_j - \frac{r}{2}\left[\sum_{k=0}^{N-j-1}(I\times F)\left(\phi_{N-j-k-1}+ +\phi_{N-j-k}\right) + \sum_{k=0}^{j-1}(I\circ F)\left(\phi_{j-k-1}+\phi_{j-k}\right)\right] = = vec\left(K_0^T W\left(\frac{r}{2}\sum_{k=0}^{j-1}\left(V_{j-k-1}+V_{j-k}\right)F - V_j\right)\right).$$

- For $j=N$

$$(I\circ I)\,\phi_N - \frac{r}{2}\sum_{k=0}^{j-1}(I\circ F)\left(\phi_{j-k-1}+\phi_{j-k}\right) = = vec\left(K_0^T W\left(\frac{r}{2}\sum_{k=0}^{j-1}\left(V_{j-k-1}+V_{j-k}\right)F - V_N\right)\right),$$

where $\phi_j = vec(\Phi_j), j = 1, 2, \ldots, N$.

4.3 Approximate Lyapunov matrix

With the piecewise linear approximation of the initial function in our hands, we now address the problem of searching the corresponding solution of the integral delay equation (4). Indeed, the problem can be formulated as the following initial value problem:

$$U(\tau) = \left(\int_{-h}^{0} U(\tau+\theta)d\theta\right) F, \quad \tau \geq 0, \tag{20}$$

$$U(\tau) = \hat{\Phi}(\tau), \quad \tau \in [-h, 0]. \tag{21}$$

Note that such a problem cannot be solved by means of the well-known step by step method for constructing solutions of differential delay systems Bellman and Cooke (1963). Here, we propose a method for solving the initial value problem (20)-(21) and that indeed it can be used to numerically construct solutions of integral delay systems of the form (1).

From (20) we have for $\tau \in [0, h]$

$$U(\tau) = W(\tau) + \left(\int_{0}^{\tau} U(\xi)d\xi\right) F, \tag{22}$$

where

$$W(\tau) = \left(\int_{\tau-h}^{0} \hat{\Phi}(\xi)d\xi\right) F.$$

It follows that the initial value problem (20)-(21) is equivalent to the problem of finding a solution of the integral equation (22). The equation (22) is a Volterra type equation for which the existence and uniqueness of solutions can be guaranteed by means of operator theory or successive approximations. In particular, the method of successive approximations provides us a procedure to numerically compute solutions of (22). More precisely, let $U_0(\tau) = W(\tau)$ and $U_j(\tau), j = 1, 2, 3, \ldots,$ be the sequence determined inductively as follows:

$$U_j(\tau) = W(\tau) + \left(\int_{0}^{\tau} U_{j-1}(\xi)d\xi\right) F, \quad \tau \in [0, h].$$

It is possible to prove that $U_j(\tau)$ converges (uniformly on τ) to a limit matrix $U(\tau)$ when $j \to \infty$ and that this matrix $U(\tau)$ satisfies the integral equation (22). The sequence $U_j(\tau)$ provides a procedure to numerically construct an approximation of $U(\tau)$ for each $\tau \in [0, h]$. Note that by continuing this process on intervals of length h one can obtain the solution $U(\tau)$ for any $\tau \geq 0$.

Thus, by using this procedure, let $\hat{U}(\tau) = \hat{U}\left(\tau, \hat{\Phi}\right), \tau \in [0, h]$, be the approximate solution of the dynamic equation (4) corresponding to the initial condition $\hat{\Phi}$.

The matrix $\hat{U}(\tau)$ for $\tau \in [-h, h]$ is then the desired approximate Lyapunov matrix.

5. APPROXIMATION ERROR

In this section, we present an approach for evaluating the quality of the computed approximate Lyapunov matrix.

Firstly, we note that by the construction procedure in subsection 4.3 it follows that the matrix $\hat{U}(\tau)$ satisfies the dynamic equation (4) for $\tau \in [0, h]$ and $\hat{U}(0) = \hat{\Phi}(0)$. On the other hand, the matrix $\hat{U}(\tau)$ does not necessarily satisfy the symmetry property (5) and, moreover, the algebraic property (6) since this property has not been involved in the construction procedure, see the Remark 2.

Let us introduce the matrices

$$\Delta S(\tau) = \hat{U}(\tau) - \hat{\Phi}^T(-\tau) - K_0^T W V(\tau),$$

and

$$\Delta A = \left[\hat{U}(0) - \hat{U}^T(h) + V^T(h) W K_0\right] F + K^T(0) W K(0) + F^T \left[\hat{U}(0) - \hat{U}^T(h) + V^T(h) W K_0\right]^T.$$

The matrix $\Delta S(\tau)$ measures the error of the symmetry property while the matrix ΔA evaluates the violation of the algebraic property.

Let us substitute matrix $U(\tau)$ in the functional (3) by the approximate matrix $\hat{U}(\tau)$ and denote the new functional by $\hat{v}(x_t)$. The time derivative of $\hat{v}(x_t)$ along the trajectories of the system (1) is

$$\begin{aligned}\frac{d}{dt}\hat{v}(x_t) = &-w(x_t) + x^T(t)\Delta A x(t) \\ &+ x^T(t-h)F^T \int_{-h}^{0} \Delta S^T(\theta+h) F x(t+\theta)d\theta \\ &+ x^T(t)F^T \int_{-h}^{0} \Delta S(-\theta) F x(t+\theta)d\theta.\end{aligned} \tag{23}$$

Similar to the approach for differential delay systems in Garcia-Lozano and Kharitonov (2006) we propose to evaluate the approximation error of the computed Lyapunov matrix $\hat{U}(\tau)$ by comparing the time derivative of the functional $\hat{v}(x_t)$ with that of the functional $v(x_t)$.

By direct calculations derived from (23) we arrive at

$$\left|\frac{d}{dt}\hat{v}(x_t) - \frac{d}{dt}v(x_t)\right| \leq \alpha \left\|z(t-h)\right\|^2 + \gamma \int_{-h}^{0} \left\|z(t+\theta)\right\|^2 d\theta,$$

where

$$\alpha = \frac{\sigma}{2}\left\|F\right\|^2 \text{ and } \gamma = h\left\|F\right\|^2\left(\delta + \sigma\left\|F\right\| + \frac{\sigma}{2}\right),$$

with

$$\sigma = \max_{\tau\in[0,h]} \left\|\Delta S(\tau)\right\| \text{ and } \delta = \left\|\Delta A\right\|.$$

Observe that if $\lambda_{\min}(W_0) > \alpha$ and $\lambda_{\min}(W_1) > \gamma$ then $\frac{d}{dt}\hat{v}(x_t)$ remains negative under the approximation errors. Furthermore, if $\sigma, \delta \to 0$ then $\alpha, \gamma \to 0$ and, therefore, $\frac{d}{dt}\hat{v}(x_t) \to \frac{d}{dt}v(x_t)$. Hence, the quantity

$$\varepsilon = \max\left\{\frac{\alpha}{\lambda_{\min}(W_0)}, \frac{\gamma}{\lambda_{\min}(W_1)}\right\},$$

can be used as a measure to evaluate the quality of the approximation of the Lyapunov matrix whereas a small ε implies a better approximation.

6. NUMERICAL EXAMPLE

To illustrate the obtained results we consider the integral delay system (1) with $h = 1$ and

$$F = \begin{pmatrix} 0.25 & 0.7 \\ -0.7 & -1 \end{pmatrix}.$$

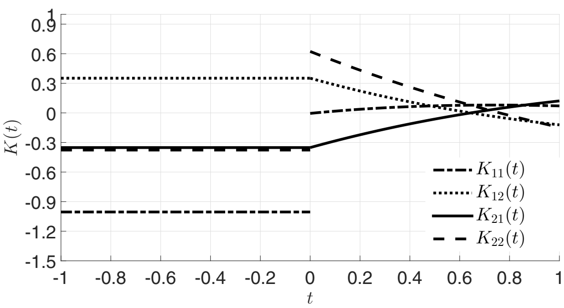

Fig. 1. Components of the fundamental matrix $K(t)$.

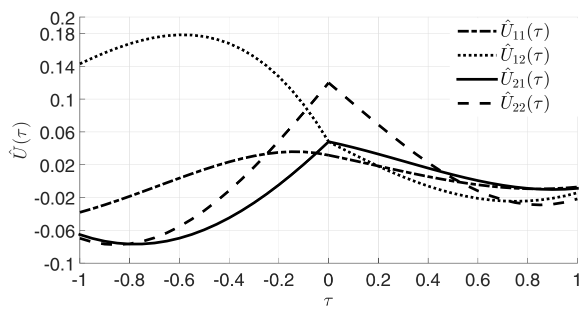

Fig. 2. Components of the approximated Lyapunov matrix $\hat{U}(\tau)$.

Since the eigenvalues of F lie in the open domain Γ whose boundary admits the parametrization

$$\partial\Gamma = \left\{ \frac{\omega \sin(\omega)}{2\left[1-\cos(\omega)\right]} + i\frac{\omega}{2} \middle| \omega \in (-2\pi, 2\pi) \right\}$$

then the system (1) is exponentially stable, see Kharitonov and Melchor-Aguilar (2000).

Note that in order to solve the system of matrix equations (17)-(19) one has to calculate the matrices $V_j, j = 1, 2, \ldots, N$, defined by (11) and, therefore, to compute the fundamental matrix $K(t)$ for $t \in [0,1]$. Since the initial condition for the fundamental matrix is known one can apply the method presented in subsection 4.3 and construct $K(t)$ for $t \in [0,1]$, see Fig. 1.

Now for $W = I$ and $N = 20$ we use the proposed algorithm to compute the approximate initial function and the corresponding approximate solution of the matrix integral delay equation (4). The approximated Lyapunov matrix $\hat{U}(\tau), \tau \in [-1,1]$, is plotted in Fig. 2.

As it can be seen from the Figs. 1 and 2, the fundamental matrix $K(t)$ presents a jump discontinuity at $t = 0$ while the Lyapunov matrix is continuous for all $\tau \in [-1,1]$ as expected from the Remark 1.

In order to evaluate the quality of the approximate Lyapunov matrix let us select matrices $W_0 = 0.15I$ and $W_1 = 0.85I$ then $W = W_0 + hW_1 = I$. Simple calculations show that $\sigma = 3.6466 \times 10^{-4}$ and $\delta = 3.3425 \times 10^{-4}$ which lead to $\alpha = 3.6719 \times 10^{-4}$ and $\gamma = 0.0021$. Then, the measure of the quality of the approximation is $\varepsilon = 0.0025$.

7. CONCLUSIONS

In this paper, we addressed the problem of computing Lyapunov matrices for integral delay systems. After showing that the existing numerical procedures of calculating Lyapunov matrices for differential delay systems cannot be applied to integral delay systems, a numerical algorithm for computing piecewise linear approximations of Lyapunov matrices is introduced.

It is important to mention that the proposed algorithm does not involve the algebraic property of the Lyapunov matrix in contrast with the methods for differential delay systems which require such a property. A measure of the approximation error is also proposed. Extensions of this work to more general classes of integral delay systems deserve further study.

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