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Distributed control for consensus on leader-followers proximity graphs

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Abstract: We propose two distributed controller solutions to the leader-followers consensus problem on inertial multiagent systems with guarantee connectivity preservation based on artificial potential functions. On the first one, we consider a virtual leader with constant velocity, in this case consensus is defined as a position reference to be tracked. On the second, the leader's velocity is time-varying. In both cases, we consider that only a subset of agents have access to leader's state information. Effectiveness of proposed controllers is illustrated with numerical simulations.

Keywords: Distributed control, leader-followers consensus, multi-agent system, connectivity preservation, artificial potential functions.

1. INTRODUCTION

Distributed control of multiple agent systems (MAS) have been intensively studied on recent years [Knorn et al. (2016)]. Potential applications such as exploration of unknown areas, surveillance or search and rescue on disaster zones have attracted a lot of attention of researchers studying computer science, engineering and control systems. A MAS, consist on a group of dynamic subsystems, called agents, interacting with each other, on local neighborhoods, through communication and/or local sensing, shearing their local state, and using the collected information to update their local state according to a distributed controller [Sakurama et al. (2015)]. On practical applications, agents might represent vehicles, satellites, sensors, mobile robots and so on, which cooperate to perform a collective task [Chen et al. (2013)], where the most fundamental ones include synchronization, flocking, rendezvous and consensus.

Consensus control on MAS, means to develop distributed controllers, *i.e.* controllers that use only local information, such that the group of agents reach an agreement on their local variables. That is, the states of all agents take a common value, which is called a *consensus state*. On earlier consensus works, the agent's dynamics was considered to be single integrators. Using algebraic graph theory results, it was proved that the communications network plays a key role on stability of consensus states [Olfati and Richard (2003); Ren et al. (2005)]. Later, on the search for more realistic agent dynamics, the consensus on double integrator dynamical agents was investigated and it was proved that, even under the same conditions as for single integrator agents, the consensus state might not be reached [Ren and Atkins (2005)]. Moreover, letting the agent's dynamics be those of an *inertial agents*, causes instability on the collective behavior even under the conditions where double integrator agents achieve consensus [Lee and Spong (2006)].

An special case of the consensus problem, arises when there's a desired common state value to reach. This kind of issue is often called the *leader-followers consensus problem* [Ren (2008)]. In other words, there's a, physic or virtual, leader agent that provides the consensus state to be tracked. Most works that consider this case assume a leader with fixed point or constant velocity dynamics, the consensus problem to a leader with time-varying velocity was investigated in [Yu et al. (2010)].

The communication network is determine by the local sensing capabilities of each node, therefore it changes over time. One way to address the effect of this type of communication network on the consensus problem is to assume that the topology changes on a discrete manner and that between switches the MAS remains connected. Under these assumptions sufficient conditions for consensus of single integrator agents were given in [R. Olfati-Saber and R. M. Murray (2004)]. The time varying nature of the communication network can also be considered by letting the network topology be imposed by relative position between agents as in [Jadbabaie et al. (2002)]. One of the firsts works, taking this approach was [Vicsek et al. (1995)], where self driven particles had to align their direction of motion according to the average of neighbor's headings.

Distributed controllers have been proposed to solve flocking problem when the network topology is imposed by the relative positions between second order continuous agents in [Olfati-Saber (2006)]. However, is possible for the agents to split into different groups as the MAS evolves over time.

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Therefore, the connectivity preservation problem has been considered in many resent investigations [Zavlanos et al. (2007); Ji and Egerstedt (2007); Su et al. (2010); Sakai et al. (2017)].

In this contribution, we proposed solutions to the leader-followers consensus problem with connectivity preservation on inertial agents, where link creation on the communication network is based on the relative position of the agents and a hysteresis process. In that sense, our problem is similar to the one considered in [Zavlanos et al. (2007); Ji and Egerstedt (2007); Su et al. (2010)]. Additionally, to solve the connectivity preservation part of the problem we use a bounded artificial potential function (APF) as part of the proposed distributed controller [Su et al. (2010); Sakai et al. (2017)]. However, our results are different since we consider the case of inertial agents and virtual leader with constant velocity, which has not been considered in previous references. Moreover, unlike the works mention above, we consider that not all agents are connected to the time-varying velocity of the leader. As such, we extend previous works by considering inertial agents, which might have different inertia values; a virtual leader with fixed velocity to be tracked; and a distributed controller design that assumes that only a few agents have access to leader's states.

2. PROBLEM STATEMENT

Consider a group of N inertial agents of the form

$$\dot{p}_i = v_i, \quad m_i \dot{v}_i = u_i, \quad i = 1, \ldots, N, \tag{1}$$

where $p_i, v_i, u_i \in \mathbb{R}^n$ and $m_i \in \mathbb{R}_{>0}$ are respectively the position, velocity, control input and mass of agent i.

The considered virtual leader's dynamics

$$\dot{p}_l = v_l, \quad \dot{v}_l = f(p_l, v_l, t) \tag{2}$$

where $p_l, v_l \in \mathbb{R}^n$ are the position and velocity, respectively, and $f(\cdot,\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}^n$ is a continuous Lipschitz function.

Suppose all agents have the same sensing/influence radio r. The MAS's underlying network is represented by a proximity graph $\mathcal{G}(p) = (\mathcal{V}, \mathcal{E}(p))$, where $\mathcal{V} = \{1, \ldots, N\}$ is a fixed set of nodes and $\mathcal{E}(p) = \{(i,j) | i,j \in \mathcal{V}\}$ is a position dependent set of edges with $p = \left[p_1^T, \ldots, p_N^T\right]^T$. Also, denote by $\mathcal{N}_i = \{j | (i,j) \in \mathcal{E}(p)\}$ the neighbor set of agent's i. Clearly, since it depends on the relative position between agents, the resulting graph is time-varying. Additionally, the dynamic set of links $\mathcal{E}(p)$ evolves such that

(1) initial links are $\mathcal{E}(p(0)) = \{(i,j) | \, \|p_{ij}(0)\| < r - \epsilon\}$, for every $i, j \in \mathcal{V}$,
(2) if the link $(i,j) \notin \mathcal{E}(p(t^-))$ and $\|p_{ij}(t)\| < r - \epsilon$, then $(i,j) \in \mathcal{E}(p(t))$ and
(3) if $\|p_{ij}(t)\| \geq r$, then $(i,j) \notin \mathcal{E}(p(t))$

where $\epsilon \in (0, r)$ is a given constant. This hysteresis process is crucial for preserving connections [Zavlanos et al. (2007); Ji and Egerstedt (2007); Su et al. (2010)]. Through this paper $\|\cdot\|$ denote the Euclidean norm and $p_{ij} = p_i - p_j$ is the relative position between a pair of agents. Figure 1 illustrates how links are added or deleted from $\mathcal{E}(p)$.

Some graph related matrices are; the adjacency matrix $\mathcal{A}(\mathcal{G}) \in \mathbb{R}^{N \times N}$ where a_{ij} is it's ij-th element, defined has

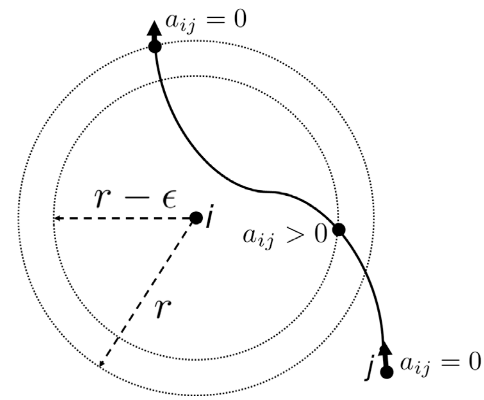

Fig. 1. Indicator function

$a_{ii} = 0$ and $a_{ij} > 0$ if $(i,j) \in \mathcal{E}(p)$; and the Laplacian matrix is $\mathcal{L}(\mathcal{G}) \in \mathbb{R}^{N \times N}$, defined has $l_{ii} = \sum_{j \neq i} a_{ij}$ and $l_{ij} = -a_{ij}$. Notice $\sum_{i=1}^{N} l_{ij} = 0$, thus $\mathcal{L}(\mathcal{G})$ is diffusive. A relevant result for Laplacian matrices, and useful for this research is the next

Lemma 1. (Su et al. (2011)). If $\mathcal{G}$ is a connected undirected graph, $\mathcal{L}$ is the symmetric Laplacian of graph $\mathcal{G}$ and the matrix $\mathcal{H} = diag(h_1, \ldots, h_N)$ with at least one positive element, then all eigenvalues of the matrix $(\mathcal{L} + \mathcal{H})$ are positive. Moreover, if $\mathcal{G}_1$ is a graph generated by adding some edge(s) into the graph $\mathcal{G}$, then $\lambda_1(\mathcal{L}_1 + \mathcal{H}) \geq \lambda_1(\mathcal{L} + \mathcal{H})$, where $\mathcal{L}_1$ is the symmetric Laplacian of the graph $\mathcal{G}_1$.

The main goal of this work is to develop distributed controllers such that a group of inertial agents follow a virtual leader. This controllers should make positions and velocities of all agents approach leader's. To achieve this goal, the given controller will have the form $u_i = \alpha_i + \beta_i + \gamma_i$, in a similar way to controllers previously reported in literature, like those in [Olfati-Saber (2006); Zavlanos et al. (2007); Su et al. (2010)]; where, α_i is a gradient-base therm to enforce neighboring agents to the same position and connectivity preservation, β_i is a consensus therm between agents, which regulates velocities difference of neighbor systems, and γ_i which consist of a consensus therm with leader's states for informed agents.

3. CONTROL ALGORITHMS

In this section we present the main results

3.1 Artificial Potential Function

The nonnegative potential function is defined such that depends on relative distances between agents $\|p_{ij}\|$, differentiable at least for $\|p_{ij}\| \in [0, r]$ and satisfying

(i) $\psi(\|p_{ij}\|) \to \bar{\psi}$ as $\|p_{ij}\| \to r$;
(ii) $\frac{\partial \psi(\|p_{ij}\|)}{\partial \|p_{ij}\|} > 0$ for $\|p_{ij}\| \in (0, r)$;
(iii) $\lim_{\|p_{ij}\| \to 0} \left(\frac{\partial \psi(\|p_{ij}\|)}{\partial \|p_{ij}\|} \frac{1}{\|p_{ij}\|} \right)$ is nonnegative and bounded.

Condition (i) states the APF is bounded. Meanwhile, condition (ii) stipulates that, the potential between pair of agents, is an increasing function of their relative distances $\|p_{ij}\|$, this makes two neighbor agents to attract each other. Finally, condition (iii), requires the gradient's

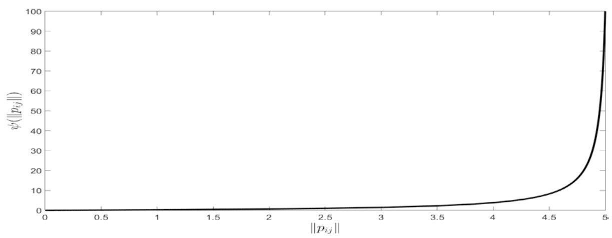

Fig. 2. Artificial potential function (3), [Su et al. (2010)].

magnitude between two agents be bounded, and take it's maximum value $\bar{\psi}$ when they are at the same position. Figure 2 shows an example of this kind of APFs, reported on [Su et al. (2010)], with equation:

$$\psi(\|p_{ij}\|) = \frac{\bar{\psi}\,\|p_{ij}\|^2}{\bar{\psi}(r - \|p_{ij}\|) + \|p_{ij}\|^2}, \tag{3}$$

with gradient respect to agent's i position as

$$\nabla_{p_i}\psi\left(\|p_{ij}\|\right) = \frac{\bar{\psi}^2\,\|p_{ij}\|\,(2r - \|p_{ij}\|)}{\left[\bar{\psi}(r - \|p_{ij}\|) + \|p_{ij}\|^2\right]^2}\frac{(p_i - p_j)}{\|p_{ij}\|}. \tag{4}$$

Clearly, APF (3) fulfills conditions (i-iii). The difference between (3) and those used other works, like [Zavlanos et al. (2007); Ji and Egerstedt (2007)], is it's boundedness. The APF presented in [Zavlanos et al. (2007)] tends to infinity when distances between neighbor agents approach the sense radio; thats why unbounded APFs are unpractical for real world implementations, because controllers might need a infinite amount of energy when to preserve connections.

3.2 Leader with constant velocity

Let leader's acceleration be $f(p_l, v_l, t) \equiv 0$. Then, the leader, might represent a fixed, or time-variable with constant velocity, reference position, depending on initial conditions of (2).

Consider the next distributed controller

$$\begin{aligned} u_i = &- \sum_{j\in\mathcal{N}_i}\nabla_{p_i}\psi\left(\|p_{ij}\|\right) - \sum_{j\in\mathcal{N}_i} a_{ij}(v_i - v_j) \\ &- h_i\left((p_i - p_l) + (v_i - v_l)\right), \quad i = 1,\ldots,N, \end{aligned} \tag{5}$$

were $h_i > 0$ if agent i receives information from the leader and $h_i = 0$ otherwise. This kind of agent is usually called an *informed agent.*

Let $\tilde{p}_i = p_i - p_l$ and $\tilde{v}_i = v_i - v_l$ be the position and velocity errors of agent i respect leader's. Also notice that, since $p_{ij} = \tilde{p}_{ij} = \tilde{p}_i - \tilde{p}_j$ and by definition of (3), the equality $\psi\left(\|p_{ij}\|\right) = \psi\left(\|\tilde{p}_{ij}\|\right)$ holds.

Define the total sum of potential and kinetic energy between every agent and the virtual leader has follows

$$V(t) = \frac{1}{2}\sum_{i=1}^{N}\left(\sum_{j\in\mathcal{N}_i}\psi\left(\|\tilde{p}_{ij}\|\right) + h_i\tilde{p}_i^T\tilde{p} + m_i\tilde{v}_i^T\tilde{v}_i\right). \tag{6}$$

Notice that, the initial energy of the complete system $V_0 = (p(0), v(0))$, is bounded by

$$V_0 \le \frac{1}{2}\sum_{i=1}^{N}\left(m_i\tilde{v}_i^T(0)\tilde{v}_i(0) + h_i\tilde{p}_i^T(0)\tilde{p}_i(0)\right) + \bar{V}_i(0) = \bar{V} \tag{7}$$

with

$$\bar{V}_i(0) = \frac{N(N-1)}{2}\psi\left(\|r - \epsilon\|\right). \tag{8}$$

Also, define the set $\Omega_0 = \left\{\tilde{p}(0), \tilde{v}(0) \in \mathbb{R}^{nN} : \bar{V} < \bar{\psi}\right\}$. This set represent all the valid initial conditions such that, for any given maximum value of the APF (3) and hysteresis value ϵ, the total initial energy of the system be smaller than the needed to break a link. This restriction will be a key point on the proof of this subsection result.

The next result states the inertial MAS tracks a leader with constant velocity.

Theorem 1. Consider a system of N inertial agents with model (1) applying controller (5) and a virtual leader with dynamics (2) with $f(p_l, v_l, t) \equiv 0$. Suppose the initial proximity graph $\mathcal{G}(p(0))$ is connected, and the initial error conditions $\tilde{p}(0), \tilde{v}(0) \in \Omega_0$, then the following results hold:

(i) $\mathcal{G}(p)$ remains connected all the time $t \ge 0$,
(ii) all agents asymptotically converge to leader's position and velocity.

Proof. First, notice that the error dynamics is

$$\dot{\tilde{p}}_i = \tilde{v}_i, \quad m_i\dot{\tilde{v}}_i = u_i, \quad i = 1,\ldots,N, \tag{9}$$

where u_i can be rewritten in therms of errors like

$$\begin{aligned} u_i = &- \sum_{j\in\mathcal{N}_i}\nabla_{\tilde{p}_i}\psi\left(\|\tilde{p}_{ij}\|\right) - \sum_{j\in\mathcal{N}_i}(\tilde{v}_i - \tilde{v}_j) \\ &- h_i\left(\tilde{p}_i + \tilde{v}_i\right), \quad i = 1,\ldots,N, \end{aligned} \tag{10}$$

Proof of part (i).

Assume the network switches every instant t_k with $k = 1, 2, \ldots$, and remains fixed over the interval $[t_{k-1}, t_k)$; also name the initial time $t_0 = 0$. Taking the time derivative of (6) over interval $[t_0, t_1)$ yields

$$\dot{V}(t) = \sum_{i=1}^{N}\left(\frac{1}{2}\sum_{j\in\mathcal{N}_i}\dot{\psi}\left(\|\tilde{p}_{ij}\|\right) + h_i\dot{\tilde{p}}_i^T\tilde{p}_i + m_i\tilde{v}_i^T\dot{\tilde{v}}_i\right), \tag{11}$$

with therms

$$\begin{aligned} \frac{1}{2}\sum_{j\in\mathcal{N}_i}\dot{\psi}\left(\|\tilde{p}_{ij}\|\right) &= \tilde{v}_i^T\sum_{j\in\mathcal{N}_i}\nabla_{\tilde{p}_i}\psi\left(\|\tilde{p}_{ij}\|\right), \\ h_i\dot{\tilde{p}}_i^T\tilde{p}_i &= h_i\tilde{v}_i^T\tilde{p}_i, \\ m_i\tilde{v}_i^T\dot{\tilde{v}}_i &= \tilde{v}_i^T u_i. \end{aligned}$$

Thus, rewriting equation (11) we have

$$\begin{aligned} \dot{V}(t) = &\sum_{i=1}^{N}\tilde{v}_i^T\sum_{j\in\mathcal{N}_i}\nabla_{\tilde{p}_i}\psi\left(\|\tilde{p}_{ij}\|\right) + \sum_{i=1}^{N}h_i\tilde{v}_i^T\tilde{p}_i \\ &- \sum_{i=1}^{N}\tilde{v}_i^T\sum_{j\in\mathcal{N}_i}\nabla_{\tilde{p}_i}\psi\left(\|\tilde{p}_{ij}\|\right) - \sum_{i=1}^{N}h_i\tilde{v}_i^T\tilde{p}_i \\ &- \sum_{i=1}^{N}\tilde{v}_i^T\sum_{j\in\mathcal{N}_i}a_{ij}(\tilde{v}_i - \tilde{v}_j) - \sum_{i=1}^{N}h_i\tilde{v}_i^T\tilde{v}_i \\ = &-\tilde{v}^T\left(\mathcal{L}_{\mathcal{H}} \otimes I_n\right)\tilde{v} \le 0 \end{aligned} \tag{12}$$

where $\mathcal{L}_{\mathcal{H}} = \mathcal{L} + \mathcal{H}$ with $\mathcal{H} = diag(h_1, \ldots, h_N)$. Equation (12), implies that $V(t) \le V_0 \le \bar{V}$ for $t \in [t_0, t_1)$, also since initial conditions $p(0), v(0) \in \Omega_0$, then $V(t) \le \bar{V} < \bar{\psi}$. The latter, implies that there are no edge distances that tends to r for $t \in [t_0, t_1)$. Since no edges are lost before t_1, new edges must be added to the network on that switching instant. Without lost of generality, assume there are $0 < q_1 \le \frac{(N-1)(N-2)}{2}$ new edges on the network, thus $V(t_1) \le V_0 + q_1\psi(\|r-\epsilon\|) \le \bar{V}$. Similar to the aforementioned analysis, the time derivative of (6) over time interval $[t_{k-1}, t_k)$ is

$$\dot{V}(t) = -\tilde{v}^T(\mathcal{L}_{\mathcal{H}} \otimes I_n)\tilde{v} \le 0 \tag{13}$$

which implies $V(t) \le \bar{V} < \bar{\psi}$, thus there are no edge distances that will tend to r and no edges will be lost on interval $[t_{k-1}, t_k)$ for any $k = 1, 2, \ldots$. Since $\mathcal{G}(p(0))$ is connected and no edges on $\mathcal{E}(p(0))$ will be lost, then $\mathcal{G}(p)$ remains connected for any time $t \ge 0$.

Proof of part (ii).

Assume there are q_k new edges added to the network on instant t_k. Clearly, $0 < q_k \le \frac{(N-1)(N-2)}{2}$ thus, from equation (13), we have $V(t_k) \le V_0 + (q_1 + \ldots + q_k)\psi(\|r-\epsilon\|) \le \bar{V}$. Therefore, for system (1) with distributed controller (5), the number of network switches is finite, which implies the graph $\mathcal{G}(p)$ eventually become fixed. Hence, the remaining analysis is restricted for time interval $[t_k, \infty)$. Notice, the distance between agents is no longer than $\psi^{-1}(\bar{V})$. Then, the set

$$\Omega = \left\{\hat{\tilde{p}} \in D_{\mathcal{G}}, \tilde{v} \in \mathbb{R}^{nN} : V(\hat{\tilde{p}}, \tilde{v}) \le \bar{V}\right\}, \tag{14}$$

is positively invariant, where

$$D_{\mathcal{G}} = \left\{\hat{\tilde{p}} \in \mathbb{R}^{nN^2} : \|\tilde{p}_{ij}\| \in \left[0, \psi^{-1}(\bar{V})\right], \forall (i,j) \in \mathcal{E}(p)\right\},$$

and $\hat{\tilde{p}} = \left[\tilde{p}_{11}^T, \ldots, \tilde{p}_{1N}^T, \ldots, \tilde{p}_{N1}^T, \ldots, \tilde{p}_{NN}^T\right]^T$. Its clear that, since $V_0 \le \bar{V}$, then the initial error conditions $\tilde{p}(0), \tilde{v}(0) \in \Omega$, thus set $\Omega_0 \subseteq \Omega$.

Since $\mathcal{G}(p)$ is connected for all $t \ge 0$, then the distance $\|\tilde{p}_{ij}\| \le (N-1)r$ for all $(i,j) \in \mathcal{E}(p)$. Also, since $V(t) \le \bar{V}$, then $m_i\tilde{v}_i^T\tilde{v}_i \le 2\bar{V}$, in consequence $\|\tilde{v}_i\| \le \sqrt{\frac{2\bar{V}}{m_i}}$. The same reasoning is used for position error of an agent such that $h_i > 0$, the equation $h_i\tilde{p}_i^T\tilde{p}_i \le 2\bar{V}$ implies that $\|\tilde{p}_i\| \le \sqrt{\frac{2\bar{V}}{h_i}}$. Therefore Ω satisfying $V(t) \le \bar{V}$ is closed and bounded, hence compact. Notice that, the system (9) whit controller (10), is autonomous at least on the concerned time interval $[t_k, \infty)$. Hence, the LaSalle's invariance principle can be used to infer that, if the initial conditions lies in Ω, then all trajectories will converge to the largest invariant set inside the region $S = \left\{\hat{\tilde{p}} \in D_{\mathcal{G}}, \tilde{v} \in \mathbb{R}^{nN} : \dot{V} = 0\right\}$.

From equation (13), notice that

$$\dot{V}(t) = -\tilde{v}^T(\mathcal{L} \otimes I_n)\tilde{v} - \tilde{v}^T(\mathcal{H} \otimes I_n)\tilde{v} = 0$$

which implies that $\tilde{v}_1 = \ldots = \tilde{v}_N$ and $\tilde{v}_i = 0$ for any i such that $h_i > 0$, i.e. $v_1 = \ldots = v_N = v_l$. This also implies that in steady state $\dot{\tilde{v}}_i = 0$, which means the controller (5) is

$$\begin{aligned} u_i &= -\sum_{j\in\mathcal{N}_i} \nabla_{p_i}\psi(\|p_{ij}\|) - h_i(p_i - p_l) \\ &= -\sum_{j\in\mathcal{N}_i} \frac{\partial\psi(\|p_{ij}\|)}{\partial\|p_{ij}\|}\frac{p_i - p_j}{\|p_{ij}\|} - h_i(p_i - p_l) \\ &= -\sum_{j\in\mathcal{N}_i} \frac{\partial\psi(\|\tilde{p}_{ij}\|)}{\partial\|\tilde{p}_{ij}\|}\frac{\tilde{p}_i - \tilde{p}_j}{\|\tilde{p}_{ij}\|} - h_i\tilde{p}_i = 0_n. \end{aligned} \tag{15}$$

where $0_n \in \mathbb{R}^n$ is a vector with all it's entries equal to zero. Rewriting (15) on a matrix form

$$-\left(\hat{\mathcal{L}} \otimes I_n\right)\tilde{p} - (\mathcal{H} \otimes I_n)\tilde{p} = 0_{nN} \tag{16}$$

where $0_{nN} \in \mathbb{R}^{nN}$ is a vector of zeros and $\hat{\mathcal{L}}$ is a Laplacian matrix with

$$\hat{\mathcal{L}}_{ii} = \sum_{j=1, j\ne i}^{N}\left(\frac{\partial\psi(\|\tilde{p}_{ij}\|)}{\partial\|\tilde{p}_{ij}\|}\frac{1}{\|\tilde{p}_{ij}\|}\right)$$

and

$$\hat{\mathcal{L}}_{ij} = -\frac{\partial\psi(\|\tilde{p}_{ij}\|)}{\partial\|\tilde{p}_{ij}\|}\frac{1}{\|\tilde{p}_{ij}\|} \quad \text{for } i \ne j.$$

Multiplying equation (16) by $\tilde{p}^T$ takes the form

$$-\tilde{p}^T\left(\hat{\mathcal{L}} \otimes I_n\right)\tilde{p} - \tilde{p}^T(\mathcal{H} \otimes I_n)\tilde{p} = 0$$

which implies that $\tilde{p}_1 = \ldots = \tilde{p}_N$ and $\tilde{p}_i = 0$ for any i such that $h_i > 0$, i.e. $p_1 = \ldots = p_N = p_l$.

Since, $\dot{V} = 0$ only when $p_i = p_l$ and $v_i = v_l$ for all $i = 1, \ldots, N$, then the position and velocity errors of the full system are asymptotically stable. Thus, all agents track leader's position and velocity. This completes the proof of Theorem 1.

3.3 *Leader with time-varying velocity*

In this subsection the leader represent a trajectory on the state space to be followed by the inertial agents.

Consider the next controller

$$\begin{aligned} u_i = &-\frac{1}{\eta_i}\sum_{j\in\mathcal{N}_i}\nabla_{p_i}\psi(\|p_{ij}\|) - \frac{1}{\eta_i}\sum_{j\in\mathcal{N}_i}a_{ij}(v_i - v_j) \\ &+ \frac{1}{\eta_i}\sum_{j\in\mathcal{N}_i}a_{ij}\dot{v}_j - \frac{h_i}{\eta_i}\left((p_i - p_l) + (v_i - v_l) - \dot{v}_l\right), \\ &i = 1, \ldots, N \end{aligned} \tag{17}$$

where $\eta_i = \frac{1}{m_i}\left(h_i + \sum_{j\in\mathcal{N}_i}a_{ij}\right)$, which for connected networks is always positive. Notice the controller (17) uses measurements of acceleration since, in practical implementations, the velocity's derivative can be calculated by numerical differentiation [Ren and Beard (2008)] or, in some cases (*e.g.* mobile robots), directly measured by accelerometers.

Define the next function

$$W(t) = \frac{1}{2}\sum_{i=1}^{N}\left(\sum_{j\in\mathcal{N}_i}\psi(\|\tilde{p}_{ij}\|) + h_i\tilde{p}_i^T\tilde{p}_i\right) + W_{\mathcal{G}} \tag{18}$$

where

$$W_{\mathcal{G}} = \frac{1}{2}\tilde{v}^T(\mathcal{L}_{\mathcal{H}} \otimes I_n)\tilde{v}. \tag{19}$$

The initial energy of the complete system is $W_0 = W(\tilde{p}(0), \tilde{v}(0))$ bounded on the next way

$$\begin{aligned} W_0 \le &\frac{N(N-1)}{2}\psi(\|r-\epsilon\|) + \frac{1}{2}\sum_{i=1}^{N}h_i\tilde{p}_i^T(0)\tilde{p}_i(0) \\ &+ W_{\mathcal{G}}(0) = \bar{W} \end{aligned} \tag{20}$$

with

$$W_{\mathcal{G}}(0) = \frac{1}{2}\tilde{v}^T(0)(\mathcal{L}_{\mathcal{H}} \otimes I_n)\tilde{v}(0). \tag{21}$$

As in previous subsection, define the initial conditions set $\Omega_0 = \left\{\tilde{p}(0), \tilde{v}(0) \in \mathbb{R}^{nN} : \bar{W} < \bar{\psi}\right\}$.

The next result states the group of inertial agents follows a virtual leader with time-varying velocity.

Theorem 2. Consider a system of N inertial agents with model (1) applying controller (17) and a virtual leader with dynamics (2). Suppose the initial proximity graph $\mathcal{G}(p(0))$ is connected, and the initial error conditions $\tilde{p}(0), \tilde{v}(0) \in \Omega_0$, then the following results hold:

(i) $\mathcal{G}(p)$ remains connected all the time $t \geq 0$,
(ii) all agents asymptotically converge to leader's position and velocity.

Proof. The error dynamics is

$$\dot{\tilde{p}}_i = \tilde{v}_i, \quad m_i\dot{\tilde{v}}_i = u_i - m_i\dot{v}_l, \quad i = 1, \ldots, N, \tag{22}$$

where controller (17) can be rewritten in therms of error states like

$$u_i = -\frac{1}{\eta_i}\sum_{j\in\mathcal{N}_i} \nabla_{\tilde{p}_i}\psi\left(\|\tilde{p}_{ij}\|\right) - \frac{1}{\eta_i}\sum_{j\in\mathcal{N}_i} a_{ij}(\tilde{v}_i - \tilde{v}_j) + \frac{1}{\eta_i}\sum_{j\in\mathcal{N}_i} a_{ij}\dot{v}_j - \frac{h_i}{\eta_i}\left(\tilde{p}_i + \tilde{v}_i - \dot{v}_l\right), \quad i = 1, \ldots, N. \tag{23}$$

After some manipulations, error dynamics (22) with controller (23), results on

$$\begin{aligned} \dot{\tilde{p}}_i =& \tilde{v}_i, \\ \sum_{j\in\mathcal{N}_i} a_{ij}(\dot{\tilde{v}}_i - \dot{\tilde{v}}_j) + h_i\dot{\tilde{v}}_i =& -\sum_{j\in\mathcal{N}_i} \nabla_{\tilde{p}_i}\psi\left(\|\tilde{p}_{ij}\|\right) - h_i\tilde{p}_i \\ & - \sum_{j\in\mathcal{N}_i} a_{ij}(\tilde{v}_i - \tilde{v}_j) - h_i\tilde{v}_i. \end{aligned}$$

Rewriting last equation on a more compact form we have

$$\begin{aligned} \dot{\tilde{p}} &= \tilde{v}, \\ \left(\mathcal{L}_\mathcal{H} \otimes I_n\right)\dot{\tilde{v}} &= -\left(\hat{\mathcal{L}}_\mathcal{H} \otimes I_n\right)\tilde{p} - \left(\mathcal{L}_\mathcal{H} \otimes I_n\right)\tilde{v}. \end{aligned} \tag{24}$$

Proof of part (i).

Similar to proof for part (i) of Theorem 1, time derivative of (18) for every time interval $[t_{k-1}, t_k)$ is

$$\begin{aligned} \dot{W}(t) &= \frac{1}{2}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} \dot{\psi}(\|\tilde{p}_{ij}\|) + \sum_{i=1}^{N} h_i\dot{\tilde{p}}_i^T\tilde{p}_i + \tilde{v}^T\left(\mathcal{L}_\mathcal{H} \otimes I_n\right)\dot{\tilde{v}} \\ &= -\tilde{v}^T\left(\mathcal{L}_\mathcal{H} \otimes I_n\right)\tilde{v} \leq 0 \end{aligned} \tag{25}$$

this implies that $W(t) \leq \bar{W} < \bar{\psi}$, therefore, no distance edges will tend to r and no edges will be lost on time interval $[t_{k-1}, t_k)$ for $k = 1, 2, \ldots$. Since $\mathcal{G}(p(0))$ is connected and no edges on $\mathcal{E}(p(0))$ will be lost, then $\mathcal{G}(p)$ remains connected for any time $t \geq 0$.

Proof of part (ii).

Following the sames steps has in Theorem's 1 part (ii) proof, the number of network switches is finite, which implies the graph eventually becomes fixed. Thus, the analysis can be restricted on time interval $[t_k, \infty)$. Again, the set

$$\Omega = \left\{\hat{p} \in D_\mathcal{G}, \tilde{v} \in \mathbb{R}^{nN} : W(\hat{p}, \tilde{v}) \leq \bar{W}\right\} \tag{26}$$

is compact. Since $\tilde{v}^T\left(\mathcal{L}_\mathcal{H} \otimes I_n\right)\tilde{v} \leq 2\bar{W}$, the last equation along with Lemma 1, implies that $\|\tilde{v}\| \leq \sqrt{\frac{2\bar{W}}{\lambda_1(\mathcal{L}_\mathcal{H})}}$ where $0 < \lambda_1(\mathcal{L}_\mathcal{H}) \leq \ldots \leq \lambda_N(\mathcal{L}_\mathcal{H})$ are the eigenvalues of matrix $\mathcal{L}_\mathcal{H}$. From LaSalle's invariance principle is clear that

$$\dot{V}(t) = -\tilde{v}^T\left(\mathcal{L} \otimes I_n\right)\tilde{v} - \tilde{v}^T\left(\mathcal{H} \otimes I_n\right)\tilde{v} = 0$$

implying $\tilde{v}_1 = \ldots = \tilde{v}_N$ with $\tilde{v}_i = 0$ for any i such that $h_i > 0$, i.e. $v_1 = \ldots = v_N = v_l$. This also implies that, in steady state, $\dot{\tilde{v}}_i = 0$ for $i = 1, \ldots, N$. Multiplying equation 24) by $\tilde{p}^T$ its clear that

$$-\tilde{p}^T\left(\hat{\mathcal{L}} \otimes I_n\right)\tilde{p} - \tilde{p}^T\left(\mathcal{H} \otimes I_n\right)\tilde{p} = 0,$$

which implies that $\tilde{p}_1 = \ldots = \tilde{p}_N$ with $\tilde{p}_i = 0$ for any i such that $h_i > 0$, i.e. $p_1 = \ldots = p_N = p_l$.

Since $\dot{W} = 0$ only when $p_i = p_l$ and $v_i = v_l$ for all $i = 1, \ldots, N$, then position and velocity errors of the full system are asymptotically stable at zero. This completes the proof of Theorem 2.

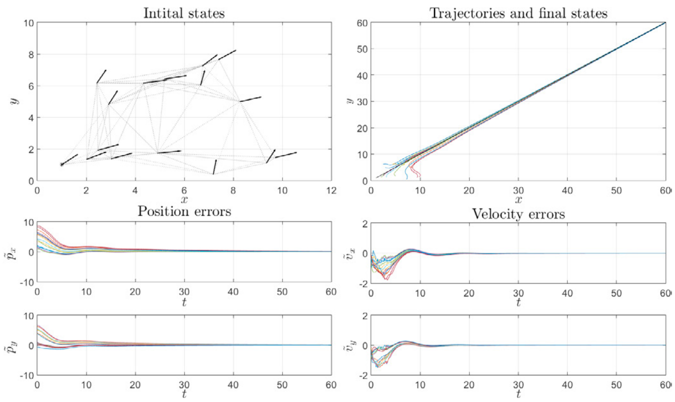

Fig. 3. Leader-followers consensus of $N = 15$ inertial agents with controller (5).

4. SIMULATIONS

4.1 Leader with constant velocity

On this simulation we considers a MAS composed of $N = 15$ inertial agents, moving in a $n = 2$ dimensional space, with masses $m_i = i$ and applying controller (5). The sensing radius for all agents is $r = 5$. Also, the maximum value of the APF (3) is $\bar{\psi} = 1000$. Respect to the hysteresis function $\epsilon = 1$. Initial positions and velocities are chosen randomly from boxes $[0, 10] \times [0, 10]$ and $[0, 1] \times [0, 1]$, respectively. Valid initial conditions are such that $V_0 \leq \bar{V}$ and $\mathcal{G}(p(0))$ is connected, both conditions are ensured before the simulation starts. Also, $h_i > 0$ for those agents who $\|\tilde{p}_i(0)\| < r$.

Figure 3 shows initial states, where solid arrows represent magnitude and direction of velocity vectors, dotted lines represents the existence of communication links between pair of neighbor agents. The solid arrow with a star on it's tail, represents magnitude and direction of leader's velocity, and dotted lines between position of the leader and an any agent, means its an informed agent. For this case, the leader is a time varying position agent with constant velocity, since $p_l(0) = [1, 1]^T$ and $v_l(0) = [1, 1]^T$. Is clear from Figure 3 that the inertial agents follow leader's position and velocity, since errors asymptotically converge to zero.

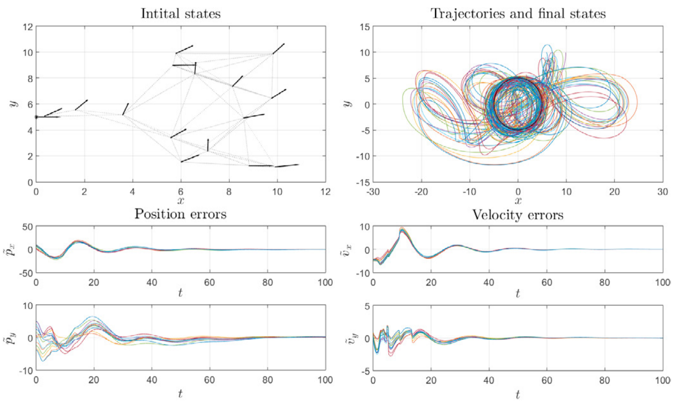

Fig. 4. Leader-followers consensus of $N = 15$ inertial agents with controller (17).

4.2 Leader time-varying velocity

For this case, we consider a group of $N = 15$ two-dimensional ($n = 2$) followers with inertias $m_i = i$ and applying controller (17). Similarly to last example, the sensing radius of all agents is $r = 5$, APF's (3) maximum value is $\bar{\psi} = 1000$ and $\epsilon = 1$. All initial positions and velocities are chosen randomly from boxes $[0, 10] \times [0, 10]$ and $[0, 1] \times [0, 1]$, respectively. Since valid initial conditions are such that $W_0 \leq \bar{W}$ and $\mathcal{G}(p(0))$ is connected, both conditions are ensured before simulation initiates. Also, $h_i > 0$ for those agents who $\|\tilde{p}_i(0)\| < r$.

Figure 4, shows a leader with acceleration $f(p_l, v_l, t) = [-5\sin(t), -5\cos(t)]^T$ such that moves on a circle of magnitude 5. Initial states are represented has in the previous example. Clearly, even when the leader moves at a time-varying velocity, the group of inertial agents converge to leader's states, since errors tends to zero has Figure 4 shows.

5. CONCLUSIONS

This paper investigates leader-followers consensus problem on multiple inertial agents with proximity networks. The distributed controllers presented use an hysteresis function for adding new links between agents, an idea adopted from [Zavlanos et al. (2007); Ji and Egerstedt (2007)], and APFs to ensure connectivity preservation. Two cases for the leader-followers consensus problem are studied; first, for a leader with constant velocity; and second, with time-varying velocity. A key difference with [Su et al. (2010)] is that, with the proposed controller, agent's positions also track leader's. In the second case, we also assumed that an informed agent has the acceleration of the leader. With respect to the result presented in [Su et al. (2010)], the main difference is that our propose controller does not need all agents to know leader's acceleration. Future works will consider obstacle avoidance, for cases where exists restrictions on the environment, and different sensing radio for each agent.

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