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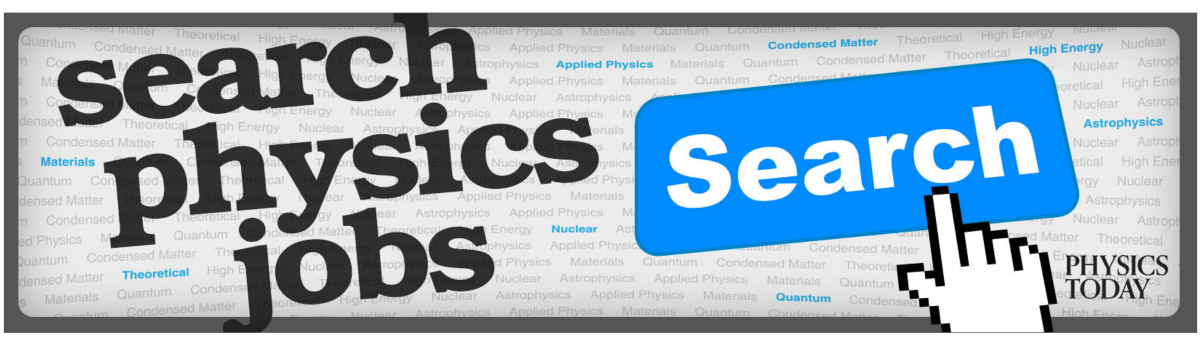

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Shape invariance through Crum transformation

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We show in a rigorous way that Crum's result regarding the equal eigenvalue spectrum of Sturm-Liouville problems can be obtained iteratively by successive Darboux transformations. Furthermore, it can be shown that all neighboring Darboux-transformed potentials of higher order, u_k and u_{k+1}, satisfy the condition of shape invariance provided the original potential u does so. Based on this result, we prove that under the condition of shape invariance, the nth iteration of the original Sturm-Liouville problem defined solely through the shape invariance is equal to the nth Crum transformation. © *2006 American Institute of Physics.*
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I. INTRODUCTION

Supersymmetric quantum mechanics,[1,2] the factorization method,[3] the Darboux transformation,[4] Crum's generalization of the former results,[5] the isospectral Hamiltonians based on the Gelfand-Levitan equation[6–9] or the Marchenko equation,[10,11] and the shape invariance condition on the potentials[12] together with a transformation defined through this condition have been in the last two decade an active area of mathematical physics[13–18] and pure mathematics.[19–21] The main concern of these areas has been the construction of isospectral Schrödinger operators and the analytical solvability of the Sturm-Liouville problem. The field allowed a deeper insight into the eigenvalue problem and served as a source for many new ideas and generalizations.[22–25] Indeed, it is almost impossible to quote all research papers on the subject (suffices to note that one review[26] and several books have been devoted to the subject[27–31]). The applications range from constructing new solvable potentials in quantum mechanics, differential equations,[32] atomic physics,[33] nuclear physics,[34] classical mechanics,[35] acoustic spectral problems[36] to quantum gravitation,[37,38] and neutrino oscillation[39] to mention a few important areas.

Mathematically, not all these transformations mentioned above are equal, or at least this is not apparent at first sight. For instance, the usual Darboux transformation is not the most general solution of the Riccati equation and as such does not give us the most general transformation in connection with the isospectral eigenvalue spectrum. On the other hand, the generalization of the Darboux transformations, namely, the so-called Crum transformation appears to be much more complicated than the original Darboux result and as such seems to offer us new avenues to construct new potentials. The third transformation of a Hamiltonian which we have in mind [defined here in Eq. (90)] is closely related to the condition of shape invariance. Hence, without doubt, there is some need to at least classify these transformations according to the complexity or generality and to uncover their relations between them. One such result in this direction is the nonequivalence of the Abraham-Moses[7] and Darboux constructions shown in Ref. 9. Two remarks are in order here. Firstly, it is understood that unlike the Darboux transformation, any transforma-

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tion in connection with the shape invariance is, of course, limited to the set of shape invariant Hamiltonians. Secondly, for completeness it is worth noting that the level of complexity of isospectral quantum systems can be increased by considering nonlinear and higher order supersymmetric transformations.[40–43] These are transformations which cannot be reached by iterative Darboux transformations. In this work, however, we will not consider these kinds of transformations and restrict ourselves to the Darboux case. After some preparatory statements we will show that the undertaking to uncover relationships between the transformations gives a simple result, namely, allowing the use of higher order Darboux transformations, we can state that all three transforms of the original Sturm-Liouville problem are equal. This result is based on a theorem which we prove in the present paper concerning higher order Darboux transformations of shape invariant potentials denoted by $u^D[k]$. The theorem states that provided the original potential satisfies the shape invariance conditions, all pairs $u^D[k]$, $u^D[k+1]$ are also mutually shape invariant. The theorem can be proved by induction. Interestingly, it intertwines this induction with another statement, this time for the wave functions. We illustrate the theorems by two examples.

II. CRUM'S RESULT

In this section, we briefly present Crum's result and comment on one identity on which Crum's result is partly based. This identity is crucial for the subsequent results which we will elaborate upon in the next section.

Let

$$W_k \equiv W(\psi_1,\psi_2,\ldots,\psi_k) = \det A, \quad A_{ij} = \frac{d^{i-1}\psi_j}{dx^{i-1}}, \quad i,j = 1,2,\ldots,k \tag{1}$$

be the Wronskian determinant of the functions $\psi_1,\psi_2,\ldots,\psi_k$, and

$$W_{k,s} = W(\psi_1,\psi_2,\ldots,\psi_k,\psi_s). \tag{2}$$

Theorem 2.1 (Crum): *If $\psi_1,\psi_2,\ldots,\psi_n$ are the solutions of the regular Sturm-Liouville problem*

$$-\frac{\mathrm{d}^2\psi_s}{\mathrm{d}x^2} + u\psi_s = \lambda_s\psi_s, \tag{3}$$

then $\psi^C[n]_s$ satisfies the Sturm-Liouville equation,

$$-\frac{\mathrm{d}^2\psi^C[n]_s}{\mathrm{d}x^2} + u^C[n]\psi^C[n]_s = \lambda_s\psi^C[n]_s, \tag{4}$$

with $\psi^C[n]_s$ and $u^C[n]_s$ given by

$$\psi_s \to \psi^C[n]_s \equiv \frac{W_{n,s}}{W_n} \tag{5}$$

and

$$u \to u^C[n] = u - 2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln W_n. \tag{6}$$

Note that the Crum transforms of ψ and u are not defined iteratively. By $[C]$ we wish to distinguish the Crum transformation from other transforms (like Darboux) which will be defined later in the text. The proof of Crum's theorem can be found in Refs. 5 and 29. We comment here only on one cornerstone of the original proof given by Crum[5] which we will also use later. The first step in the proof of Crum's result on the Wronskian determinant is to consider the derivative of W_k. Taking the derivative of

$$W_2 = \begin{vmatrix} \psi_1 & \psi_2 \\ \mathrm{d}\psi_1/\mathrm{d}x & \mathrm{d}\psi_2/\mathrm{d}x \end{vmatrix}, \tag{7}$$

we find the rather obvious result,

$$\frac{\mathrm{d}W_2}{\mathrm{d}x} = \begin{vmatrix} \psi_1 & \psi_2 \\ \psi_1'' & \psi_2'' \end{vmatrix}, \tag{8}$$

where we used the notation ψ_i'' for $\mathrm{d}^2\psi_i/\mathrm{d}x^2$, $i=1,2$. This result can be readily generalized for the $n\times n$ case.

Lemma 2.2: *For the derivative of a Wronskian determinant we have*

$$W_n' = \{\psi_1^{(n)}M_{(1,n)}^{(1)} + \psi_2^{(n)}M_{(2,n)}^{(1)} + \cdots + \psi_{n-1}^{(n)}M_{(n-1,n)}^{(1)} + \psi_n^{(n)}M_{(n,n)}^{(1)}\}. \tag{9}$$

Assume the result to be valid for $n-1$. Using the Laplace expansion according to the last line of the Wronskian W_n we get

$$\begin{aligned} W_n' = &\{\psi_1^{(n)}M_{(1,n)}^{(1)} + \psi_2^{(n)}M_{(2,n)}^{(1)} + \cdots + \psi_{n-1}^{(n)}M_{(n-1,n)}^{(1)} + \psi_n^{(n)}M_{(n,n)}^{(1)}\} \\ &+ \{\psi_1^{(n-1)}(M_{(1,n)}^{(1)})' + \cdots + \psi_{n-1}^{(n-1)}(M_{(n-1,n)}^{(1)})' + \psi_n^{(n-1)}(M_{(n,n)}^{(1)})'\}, \end{aligned} \tag{10}$$

where every $(n-1)\times(n-1)$ determinant $M_{(i,n)}^{(1)}$ is a Wronskian for which, by assumption, the theorem is valid. Hence

$$\det B \equiv \{\psi_1^{(n-1)}(M_{(1,n)}^{(1)})' + \cdots + \psi_{n-1}^{(n-1)}(M_{(n-1,n)}^{(1)})' + \psi_n^{(n-1)}(M_{(n,n)}^{(1)})'\} \tag{11}$$

is a determinant whose two last lines are equal and therefore $\det B=0$. The result [Eq. (10)] can be written as

$$W_n' = \begin{vmatrix} \psi_1 & \psi_2 & \cdots & \psi_{n-1} & \psi_n \\ \psi_1' & \psi_2' & \cdots & \psi_{n-1}' & \psi_n' \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \psi_1^{(n-2)} & \psi_2^{(n-2)} & \cdots & \psi_{n-1}^{(n-2)} & \psi_n^{(n-2)} \\ \psi_1^{(n)} & \psi_2^{(n)} & \cdots & \psi_{n-1}^{(n)} & \psi_n^{(n)} \end{vmatrix}. \tag{12}$$

We can now state a result which will be of some importance later and which is one of the important ingredients in proving Theorem 2.1 of Crum.

Lemma 2.3 (Crum): *The Wronski determinant of the two Wronskians*, W_n and $W_{n-1,s}$, *is equal to* $W_{n,s}W_{n-1}$. *In other words*

$$W(W_n, W_{n-1,s}) = W_{ns}W_{n-1}. \tag{13}$$

The proof relies on the Jacobi theorem for determinants (see the Appendix). We refer the reader to the Appendix for the proof of this Lemma too.

It is well known that for $n=1$ the Crum transformations reduce to the Darboux transformation when $W_1=\psi_1$ and $W_{1,s}=W(\psi_1,\psi_s)$. Specifically, we have

$$\psi^D[1]_s \equiv \psi^C[1]_s = \frac{W_{1,s}}{\psi_1} = \psi_s' - \frac{\psi_1'}{\psi_1}\psi_s, \quad s>1, \tag{14}$$

$$u^D[1] \equiv u^C[1] = u - 2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln W_1 = u - 2\frac{\mathrm{d}}{\mathrm{d}x}\frac{\psi_1'}{\psi_1}. \tag{15}$$

We can define higher order Darboux transformations iteratively by

$$u^D[k-1] \to u^D[k] = u[k-1] - 2\frac{\mathrm{d}}{\mathrm{d}x}\frac{(\psi^D[k-1]_k)'}{\psi^D[k-1]_k}, \qquad s > k. \tag{16}$$
$$\psi^D[k-1]_s \to \psi^D[k]_s = \frac{W(\psi^D[k-1]_k, \psi^D[k-1]_s)}{\psi^D[k-1]_k},$$

Obviously, the last equation can be written also in a way which resembles more the first Darboux transformation, i.e.,

$$\psi^D[k]_s = (\psi^D[k-1]_s)' - \frac{(\psi^D[k-1]_k)'}{\psi^D[k-1]_k}\psi^D[k-1]_s. \tag{17}$$

It is *a priori* not clear as to what connection the kth Darboux transformation has with the kth Crum transformation and if they can be related at all, except for the definition at the lowest order of Crum's transformation. The answer is provided in the next section.

III. THE CONNECTION BETWEEN HIGHER ORDER DARBOUX AND CRUM TRANSFORMATION

To this end, let us first examine the simplest case of $k=2$,

$$u^D[2] = u^D[1] - 2\frac{\mathrm{d}}{\mathrm{d}x}\frac{(\psi^D[1]_2)'}{\psi^D[1]_2}. \tag{18}$$

Since $u[D]_{[1]}$ is the Darboux transformed potential the above equation reads

$$u^D[2] = u - 2\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\psi_1'}{\psi_1} + \frac{(\psi^D[1]_2)'}{\psi^D[1]_2}\right). \tag{19}$$

According to Eq. (19), $\psi^D[1]_2 = W_{1,2}/\psi_1$ and on account of the simple identity $W_{n,n+1} = W_{n+1}$, we can write

$$u^D[2] = u - 2\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\psi_1'}{\psi_1} + \frac{(W_2/\psi_1)'}{W_2/\psi_1}\right), \tag{20}$$

which finally gives

$$u^D[2] = u - 2\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{W_2'}{W_2}\right) = u^C[2]. \tag{21}$$

Similarly, the eigenfunctions

$$\psi^D[2]_s = \frac{\begin{vmatrix} \psi^D[1]_2 & \psi^D[1]_s \\ (\mathrm{d}/\mathrm{d}x)\psi^D[1]_2 & (\mathrm{d}/\mathrm{d}x)\psi^D[1]_s \end{vmatrix}}{\psi^D[1]_2} \tag{22}$$

can be cast into the form

$$\psi^D[2]_s = \frac{1/\psi_1 \begin{vmatrix} W_2 & W_{1,s} \\ (W_2/\psi_1)' & (W_{1,s}/\psi_1)' \end{vmatrix}}{\dfrac{W_2}{\psi_1}} = \frac{\begin{vmatrix} W_2 & W_{1,s} \\ (W_2/\psi_1)' & (W_{1,s}/\psi_1)' \end{vmatrix}}{W_2}. \tag{23}$$

With the help of the standard property of determinants, namely, $\det(\mathbf{z}_1, \ldots, \mathbf{z}_i, \ldots, \mathbf{z}_n) = \det(\mathbf{z}_1, \ldots, \mathbf{z}_i + \alpha\mathbf{z}_k, \ldots, \mathbf{z}_n)$ the last equation reduces to

$$\psi^D[2]_s = \frac{1/\psi_1 \begin{vmatrix} W_2 & W_{1,s} \\ W_2' & W_{1,s}' \end{vmatrix}}{W_2}. \tag{24}$$

Applying the result of Lemma 2.3 and remembering that $W_1 = \psi_1$, one finally finds

$$\psi^D[2]_s = \frac{W_{2,s}}{W_2} = \psi^C[2]_s. \tag{25}$$

The steps above will serve as a beginning of the induction proof of the following general statement,

Theorem 3.1: *The* nth *Crum transformation is equivalent to the* nth *higher order Darboux transformation. This is to say, any Crum transformation can be reached iteratively by successive Darboux transformations, i.e.*,

$$\begin{aligned} u^C[n] &= u^D[n], \\ \psi^C[n]_s &= \psi^D[n]_s. \end{aligned} \tag{26}$$

Proof: Assuming the theorem to be valid for n means that the statement

$$\begin{aligned} u^D[n] &= u^D[n-1] - 2\frac{\mathrm{d}}{\mathrm{d}x}\frac{(\psi^D[n-1]_n)'}{\psi^D[n-1]_n}, \\ \psi^D[n]_s &= \frac{W(\psi^D[n-1]_n, \psi^D[n-1]_s)}{\psi^D[n-1]_n}, \end{aligned} \qquad s > n \tag{27}$$

is equivalent to

$$\begin{aligned} u^D[n] &= u - 2\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{W_n'}{W_n}\right), \\ \psi^D[n]_s &= \frac{W_{n,s}}{W_n}. \end{aligned} \qquad s > n. \tag{28}$$

Based on that, we have to show

$$u^C[n+1] = u^D[n+1] = u^D[n] - 2\frac{\mathrm{d}}{\mathrm{d}x}\frac{(\psi^D[n]_{n+1})'}{\psi^D[n]_{n+1}} = u - 2\frac{\mathrm{d}}{\mathrm{d}x}\frac{W_{n+1}'}{W_{n+1}} \tag{29}$$

and

$$\psi^D[n+1]_s = \frac{W(\psi^D[n]_{n+1}, \psi^D[n]_s)}{\psi^D[n]_{n+1}} = \frac{W_{n+1,s}}{W_{n+1}}. \tag{30}$$

The validity of the hypothesis of the induction for n allows us to write

$$u^D[n+1] = u^D[n] - 2\frac{d}{dx}\frac{(\psi^D[n]_{n+1})'}{\psi^D[n]_{n+1}} = u - 2\frac{d}{dx}\left(\frac{W_n'}{W_n} + \frac{(\psi^D[n]_{n+1})'}{\psi^D[n]_{n+1}}\right). \tag{31}$$

Using the validity of the hypothesis for n, but this time for the wave functions, implies

$$u^D[n+1] = u - 2\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{W_n'}{W_n} + \frac{\left(\frac{W_{n+1}}{W_n}\right)'}{\frac{W_{n+1}}{W_n}}\right) = u - 2\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{W_{n+1}'}{W_{n+1}}\right) = u^C[n+1]. \tag{32}$$

Similarly, the result for the eigenfunctions may be written as

$$\psi^D[n+1]_s = \frac{\begin{vmatrix} \psi^D[n]_{n+1} & \psi^D[n]_s \\ \psi'^D[n]_{n+1} & \psi'^D[n]_s \end{vmatrix}}{\psi^D[n]_{n+1}} = \frac{\begin{vmatrix} W_{n+1}/W_n & W_{n,s}/W_n \\ (W_{n+1}/W_n)' & (W_{n,s}/W_n)' \end{vmatrix}}{W_{n+1}/W_n}. \tag{33}$$

One easily proceeds now to verify the validity of the following equation:

$$\psi^D[n+1]_s = \frac{\begin{vmatrix} W_{n+1} & W_{n,s} \\ W_{n+1}'/W_n - W_n'W_{n+1}/W_n^2 & W_{n,s}'/W_n - W_n'W_{n,s}/W_n^2 \end{vmatrix}}{W_{n+1}} = \frac{1/W_n\begin{vmatrix} W_{n+1} & W_{n,s} \\ W_{n+1}' & W_{n,s}' \end{vmatrix}}{W_{n+1}}. \tag{34}$$

By virtue of Lemma 2.3 we can assure that

$$\psi^D[n+1]_s = \frac{W_{n+1,s}}{W_{n+1}} = \psi^C[n+1]_s \tag{35}$$

is true which completes the proof.

It is instructive to follow this theorem by an explicit example.

IV. TWO EXAMPLES

In this section we will demonstrate the above theorems by two examples. We choose first a potential which satisfies the condition of shape invariance (Morse potential) followed by the example of Ginnochio potential, which falls into the class of nonshape invariant, but solvable potentials.

Let us consider, as an example, the Sturm Liouville problem with the Morse potential, i.e.,

$$u(x;A) = 2\left[A^2 - A\left(A + \frac{\alpha}{\sqrt{2}}\right)\mathrm{sech}^2(\alpha x)\right]. \tag{36}$$

The superpartner of this potential corresponding to ψ_1 and λ_1 is

$$u^C[1] = u^D[1] = 2[A^2 - AA_1\,\mathrm{sech}^2(\alpha x)], \tag{37}$$

and the first three eigenfunctions are given by the following:

1. $\psi_1 = c_1[\mathrm{sech}(\alpha x)]^{\sqrt{2}A/\alpha}$.

2. $\psi_2 = c_2 \sinh(\alpha x)\psi_1$.

3. $\psi_3 = c_3\left(-\cosh^2(\alpha x) + \frac{(2\sqrt{2}A - \alpha)}{\alpha}\sinh^2(\alpha x)\right)\psi_1$.

In the following we will not determine the constants c_i as they are of minor importance for our results. Secondly, the results become increasingly complicated. For instance, to calculate c_1 we can use

$$\int_0^{\infty} \mathrm{sech}(ax)^{2\sqrt{2}A/\alpha}\mathrm{d}x = -\frac{1}{\sqrt{2}A}\,{}_2F_1\left(\frac{\sqrt{2}A}{\alpha},\frac{1}{2},1+\frac{\sqrt{2}A}{\alpha},\frac{\alpha}{\sqrt{2}A}\right), \tag{38}$$

where ${}_2F_1$ is the hypergeometric function. The corresponding eigenvalues can be compactly written as

$$\lambda_n = 2\left(A^2 - \left(A - \frac{(n-1)\alpha}{\sqrt{2}}\right)^2\right). \tag{39}$$

It is convenient to define A_n as

$$A_n \equiv A - \frac{n\alpha}{\sqrt{2}}, \tag{40}$$

such that the eigenvalues read now

$$\lambda_n = 2(A^2 - A_{n-1}^2), \quad n = 1,2,3, \ldots . \tag{41}$$

The first three are explicitly given as follows:

$$\lambda_1 = 0, \quad \lambda_2 = 2\sqrt{2}A\alpha - \alpha^2, \quad \lambda_3 = 4\sqrt{2}A\alpha - 4\alpha^2. \tag{42}$$

Besides Eq. (37) we will also need the following results:

$$\psi^C[1]_2 = \psi^D[1]_2 = c_2\alpha\cosh(\alpha x)\psi_1, \tag{43}$$

$$\psi^C[1]_3 = \psi^D[1]_3 = 4\sqrt{2}c_3A_1\sinh(\alpha x)\cosh(\alpha x)\psi_1. \tag{44}$$

To show explicitly the equality $u^C[2]=u^D[2]$, we start with $u^C[2]$, i.e.,

$$u^C[2] = u - 2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln W_2. \tag{45}$$

Since $W_{1,2}=W_2$ and using

$$\frac{W_{1,2}}{W_1} = c_2\alpha\cosh(\alpha x)\psi_1, \tag{46}$$

we have,

$$\ln W_2 = \ln c_2\alpha + \ln\cosh(\alpha x) + 2\ln\psi_1, \tag{47}$$

but also

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln W_2 = \alpha(\alpha - 2\sqrt{2}A)\mathrm{sech}^2(\alpha x). \tag{48}$$

Finally, with Eq. (36) we arrive at

$$u^C[2] = 2[A^2 - A_1A_2\,\mathrm{sech}^2(\alpha x)]. \tag{49}$$

Next we turn to the expression for $u^D[2]$, namely,

$$u^D[2] = u^D[1] - 2\frac{\mathrm{d}}{\mathrm{d}x}\frac{(\psi^D[1]_2)'}{\psi^D[1]_2}. \tag{50}$$

Taking into account Eq. (43) we obtain

$$\frac{(\psi^D[1]_2)'}{\psi^D[1]_2} = -\sqrt{2}A_1 \tanh(\alpha x), \tag{51}$$

and therefore

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{(\psi^D[1]_2)'}{\psi^D[1]_2} = -\sqrt{2}\alpha A_1 \,\mathrm{sech}^2(\alpha x). \tag{52}$$

From this we conclude [see Eq. (50)] that

$$u^D[2] = 2[A^2 - A_1A_2 \,\mathrm{sech}^2(\alpha x)], \tag{53}$$

and hence

$$u^C[2] = u^D[2]. \tag{54}$$

The transformed potentials here have almost identical functional form. This is, of course, due to the choice of the potential and need not be so in other cases.

To demonstrate that $\psi^C[2]_s = \psi^D[2]_s$, we calculate $\psi^C[2]_3$ to be

$$\psi^C[2]_3 = \frac{W_3}{W_2} = \frac{\begin{vmatrix} \psi_1 & \psi_2 & \psi_3 \\ \psi_1' & \psi_2' & \psi_3' \\ -\lambda_1\psi_1 & -\lambda_2\psi_2 & -\lambda_3\psi_3 \end{vmatrix}}{\begin{vmatrix} \psi_1 & \psi_2 \\ \psi_1' & \psi_2' \end{vmatrix}} = \lambda_2\psi_2\frac{W_{1,3}}{W_2} - \lambda_3\psi_3. \tag{55}$$

Making use of

$$\lambda_2\psi_2\frac{W_{1,3}}{W_2} = \left(\frac{4\sqrt{2}\lambda_2 c_3 A_1}{\alpha}\right)\sinh^2(\alpha x)\psi_1, \tag{56}$$

this becomes

$$\psi^C[2]_3 = \lambda_3 c_3 \cosh^2(\alpha x)\psi_1(x). \tag{57}$$

On the other hand

$$\psi^D[2]_3 = \frac{\begin{vmatrix} \psi[1]_2 & \psi[1]_3 \\ (\psi[1]_2)' & (\psi[1]_3)' \end{vmatrix}}{\psi[1]_2} = \frac{(\psi[1]_3/\psi[1]_2)'(\psi[1]_2)^2}{\psi[1]_2} = \left(\frac{\psi[1]_3}{\psi[1]_2}\right)'\psi[1]_2 = \lambda_3 c_3 \cosh^2(\alpha x)\psi_1, \tag{58}$$

where on the right hand side we already dropped the distinction between D and C [see Eqs. (43) and (44)]. The simple conclusion that we can draw is

$$\psi^C[2]_3 = \psi^D[2]_3. \tag{59}$$

It is instructive to consider also a case of a solvable, but nonshape invariant potential. Many such cases are known (see Refs. 44–46 and the discussion in Ref. 26) and explicit proofs that these potentials fail to satisfy the shape invariance condition were given. For instance, for the case of the Natanzon potential this was shown in Ref. 47. Many of these potentials are complicated and some, like the Natanzon case, only known in implicit form. Therefore, for the sake of efficient calculations, it is recommendable to develop first a fast algorithm to perform the desired calculations. We will do exactly that before giving the explicit example of the Ginocchio case. Imagine we would like to calculate $\psi^D[2]_3$. In turns out that the calculation can be greatly simplified by invoking the ratios $h_n=\psi_n'/\psi_n$, where ψ_n is, as usual, the eigenfunction to the ϵ_n eigenvalue. It is now a straightforward exercise to show that

$$\psi^D[2]_3 = \frac{\begin{vmatrix} \psi^D[1]_2 & \psi^D[1]_3 \\ (\psi^D[1]_2)' & (\psi^D[1])_3)' \end{vmatrix}}{(\psi^D[1]_2)} \tag{60}$$

is equal to

$$\frac{1}{(h_1-h_0)\psi_1}[(h_1-h_0)\{(h_2-h_0)'+(h_2-h_0)h_2\}\psi_1\psi_2-(h_2-h_0)\{(h_1-h_0)'+(h_1-h_0)h_1\}\psi_1\psi_2]. \tag{61}$$

Using the Schrödinger equation the latter simplifies to

$$\psi^D[2]_3 = \frac{[\epsilon_0\{h_2-h_1\}-\epsilon_1\{h_2-h_0\}+\epsilon_2\{h_1-h_0\}]}{(h_1-h_0)}\psi_2. \tag{62}$$

It is obvious that, provided we know the functions h_0, h_1, h_2, and ψ_2, this expression allows a fast calculation or the wave function $\psi^D[2]_3$ for arbitrary potential. Crum's result gives

$$\psi^C[2]_3 = \frac{W_{2,3}}{W_2} = \frac{W_3}{W_2} = \frac{W_3}{(h_1-h_0)\psi_0\psi_1}, \tag{63}$$

where W_3 is

$$W_3 = \begin{vmatrix} \psi_0 & \psi_1 & \psi_2 \\ \psi_0' & \psi_1' & \psi_2' \\ \psi_0'' & \psi_1'' & \psi_2'' \end{vmatrix} = \begin{vmatrix} \psi_0 & \psi_1 & \psi_2 \\ h_0\psi_0 & h_1\psi_1 & h_2\psi_2 \\ \psi_0'' & \psi_1'' & \psi_2'' \end{vmatrix}$$
$$= \begin{vmatrix} \psi_0 & \psi_1 & \psi_2 \\ h_0\psi_0 & h_1\psi_1 & h_2\psi_2 \\ u-\epsilon_0\psi_0 & u-\epsilon_1\psi_1 & u-\epsilon_1\psi_2 \end{vmatrix} = - \begin{vmatrix} \psi_0 & \psi_1 & \psi_2 \\ h_0\psi_0 & h_1\psi_1 & h_2\psi_2 \\ \epsilon_0\psi_0 & \epsilon_1\psi_1 & \epsilon_2\psi_2 \end{vmatrix}. \tag{64}$$

Hence, taking Eq. (63) into account, we can show that

$$\psi^C[2]_3 = \frac{[\epsilon_0\{h_2-h_1\}-\epsilon_1\{h_2-h_0\}+\epsilon_2\{h_1-h_0\}]\psi_0\psi_1\psi_2}{(h_1-h_0)\psi_1\psi_0}, \tag{65}$$

which obviously implies that $\psi^D[2]_3=\psi^C[2]_3$. This, as it stands, is a general proof for a subcase of our general theorem. On purpose above we have used different steps than in the proof of our general theorem. The idea behind it is to demonstrate that in an explicit example we would be only repeating the very same steps as above. It is therefore sufficient to calculate every time only the right hand side of Eq. (65). The equality $\psi^D[2]_3=\psi^C[2]_3$ is guaranteed by Eqs. (62) and (65). We can now apply the results for $\psi^D[2]_3$ by choosing the Ginocchio potential,

$$V(x) = \left\{-\beta^2\nu(\nu+1)+\frac{1}{4}(1-\beta^2)[5(1-\beta^2)y^4-(7-\beta^2)y^2+2]\right\}(1-y^2), \tag{66}$$

where $y(x)$ satisfies the following differential equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (1-y^2)[1-(1-\beta^2)y^2], \tag{67}$$

and β and υ are parameters.

The wave functions of this problem are known to be expressible through Gegenbauer polynomials $C_n^{(a)}(x)$, namely,

$$\psi_n = (1-\beta^2)^{\mu_n/2}[g(y)]^{-(2\mu_n+1)/4}C_n^{(\mu_n+1/2)}(f(y)), \tag{68}$$

where

$$g(y) = 1-(1-\beta^2)y^2, \quad f(y) = \frac{\beta y}{\sqrt{g(y)}}. \tag{69}$$

The value of μ_n is connected to the eigenvalue ϵ_n by $\epsilon_n = -\mu_n^2\beta^4$ such that

$$\mu_n\beta^2 = \sqrt{\beta^2(\upsilon+1/2)^2+(1-\beta^2)(n+1/2)^2}-(n+1/2). \tag{70}$$

The first four Gegenbauer polynomials are given as follows:

$$\begin{aligned} C_0^{(\mu_0+1/2)}(f(y)) &= 1, \\ C_1^{(\mu_1+1/2)}(f(y)) &= 2(\mu_1+1/2)[f(y)], \\ C_2^{(\mu_2+1/2)}(f(y)) &= 2(\mu_2+1/2)(\mu_2+3/2)[f(y)]^2-(\mu_2+1/2), \\ C_3^{(\mu_3+1/2)}(f(y)) &= \frac{4}{3}(\mu_3+1/2)(\mu_3+3/2)(\mu_3+5/2)[f(y)]^3-2(\mu_3+1/2)(\mu_3+3/2)[f(y)]. \end{aligned} \tag{71}$$

These functions can be used, in the next step, to compute explicitly the ratios $h_i = \psi_i'/\psi_i$. We obtain

$$\begin{aligned} h_0 &= \frac{\psi_0'}{\psi_0} = \frac{[g(y)]'}{g(y)} = -2(1-\beta^2)y(1-y^2), \\ h_1 &= \frac{\psi_1'}{\psi_1} = \frac{[g(y)]'}{[g(y)]} + \frac{[f(y)]'}{[f(y)]} = \frac{(1-y^2)}{y}\{1-2(1-\beta^2)y^2\}, \\ h_2 &= \frac{\psi_2'}{\psi_2} = \left(\frac{[g(y)]'}{[g(y)]} + \frac{2[f(y)][f(y)]'}{([f(y)]^2-[1/(2\mu_2+3)])}\right), \end{aligned} \tag{72}$$

where we have used

$$\frac{[f(y)]'}{[f(y)]} = \frac{1}{y}(1-y^2). \tag{73}$$

Noting that the h_i are proportional $(1-y^2)$ and that $h_1-h_0=(1-y^2)/y$, we can insert our results into Eq. (65) to obtain

$$\begin{aligned} \psi^C[2]_3 &= (1-\beta^2)^{\mu_2/2}(\mu_2+1/2)(2\mu_2+3)[g(y)]^{-(2\mu_2+1)/4} \\ &\times\left\{(\epsilon_2-\epsilon_0)\left([f(y)]^2-\frac{1}{(2\mu_2+3)}\right)-(\epsilon_1-\epsilon_0)2[f(y)]^2\right\}. \end{aligned} \tag{74}$$

Turning our attention to the potential the superpartner of V in Eq. (66) it is not difficult to see that the superpartner is given by

$$V^D[1] = V^C[1] = V - 2\frac{d^2}{dx^2}\ln W_1$$
$$= V - 2\frac{d}{dr}\frac{[g(y)]'}{g(y)} = V + 4(1-\beta^2)(1-3y^2)^2(1-y^2)[1-(1-\beta^2)y^2]. \tag{75}$$

The second Crum iteration yields

$$V^C[2] = V - 2\frac{d^2}{dx^2}\ln W_2 = V - 2[\{-2 + (1-\beta^2)[5y^2-3]\} + 10y^2(1-\beta^2)](1-y^2)[1-(1-\beta^2)y^2]. \tag{76}$$

To proof that this is equivalent to the second Darboux transformations it is convenient, as was the case with the wave functions, to provide first a general proof for this subcase. Starting with the definition, it is straightforward to show that

$$\psi^D[1]_2 = (h_1 - h_0)\psi_1, \tag{77}$$

which leads to

$$V^D[2] = V^D[1] - 2\frac{d}{dr}\frac{(\psi^D[1]_2)'}{\psi^D[1]_2} = V - 2\frac{d^2}{dr^2}\ln\psi_0 - 2\frac{d^2}{dr^2}\ln(h_1-h_0)\psi_1$$
$$= V - 2\frac{d^2}{dr^2}\{\ln(h_1-h_0)\psi_0\psi_1\} = V - 2\frac{d^2}{dr^2}\ln W_2 = V^C[2]. \tag{78}$$

In taking explicit examples we would be only repeating the very same steps as above. This demonstration concludes our two examples.

V. THE CONNECTION BETWEEN SHAPE INVARIANCE AND CRUM TRANSFORMATIONS

In view of the results of the previous section we can now drop the distinction between higher order Darboux (D) and Crum (C) transformations.

Let a denote a set of parameters in the original potential, i.e.,

$$u = u(x;a). \tag{79}$$

The condition for shape invariance of u is given by

$$u[1](x;a) = u(x;f(a)) + R(f(a)), \tag{80}$$

where $u[1](x;a)$ is the first Darboux transform of the original potential, f transforms a into another set $f(a)$, and $R(f(a))$ is a function of the parameters. In the following, we use the usual notation $a_m \equiv f^m(a)$, where m indicates the function f applied m times.

In the preceding section we established an equivalence between higher order Darboux transformation and the Crum result. Since the shape invariance is given in terms of the first order Darboux transformation, it is legitimate to ask if higher order Darboux transformations (Crum transformations) play a role in the Schrödinger equation with shape invariant potentials. As a first step we will prove the following theorem.

Lemma 5.1: *Under the condition of shape invariance one has*

$$\psi_s(x;a_1) = \psi[1]_{s+1}(x;a), \tag{81}$$

up to a multiplicative constant and

$$\lambda_s(a_1) + R(a_1) = \lambda_{s+1}(a). \tag{82}$$

In the above $\psi_s(x;a)$ *denotes the eigenfunction to the Hamiltonian with the potential* $u(x;a)$ *with the eigenvalue* λ_s.

These results are not new. But since we will make use of them, we offer here a short proof. We start with initial Sturm-Liouville problem,

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+u(x;a)\right)\psi_s(x;a)=\lambda_s(a)\psi_s(x;a) \tag{83}$$

and

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+u[1](x;a)\right)\psi[1]_s(x;a)=\lambda_s(a)\psi[1]_s(x;a), \quad s>1. \tag{84}$$

Equation (83) is valid for any a, hence we may write

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+u(x;a_1)\right)\psi_s(x;a_1)=\lambda_s(a_1)\psi_s(x;a_1), \tag{85}$$

and add $R(f(a))\psi_s(x;f(a))$ on both sides implying the following identity:

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+u(x;a_1)+R(a_1)\right)\psi_s(x;a_1)=(\lambda_s(a_1)+R(a_1))\psi_s(x;a_1). \tag{86}$$

Due to the shape invariance condition (for the sake of formulating the next Lemma we can say that $u[1]$ and u are pairwise shape invariant) this becomes

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+u[1](x;a)\right)\psi_s(x;a_1)=(\lambda_s(a_1)+R(a_1))\psi_s(x;a_1). \tag{87}$$

Without loss of generality, the spectrum can be ordered as $\lambda_1<\lambda_2<\lambda_3<\cdots$. Hence, $\{\lambda_s(a)\}$, $\{\lambda_s(a_1)\}$, and $\{\lambda_s(a_1)+R(a_1)\}$ are similarly ordered sets. $\psi[1]_{s+1}$ is then an eigenfunction to the ordered spectrum $\lambda_2<\lambda_3<\cdots$. We can conclude that up to a multiplicative factor

$$\psi_s(x;a_1)=\psi[1]_{s+1}(x,a) \tag{88}$$

and

$$\lambda_s(a_1)+R(a_1)=\lambda_{s+1}(a). \tag{89}$$

In preparation of the main theorem of this section we prove the next Lemma.

Lemma 5.2: *By virtue of the the above Lemma and under the condition that* u *and* $u[1]$ *are pairwise shape invariant*, $u[1]$ and $u[2]$ *are also pairwise shape invariant i.e.*,

$$u[2](x;a)=u[1](x;a_1)+R(a_1). \tag{90}$$

The proof can be done in two steps.

1. The condition of shape invariance and the definition of the Darboux transformation allows us to write

$$u(x;a)-2\frac{\mathrm{d}}{\mathrm{d}x}\frac{\psi_1'(x;a)}{\psi_1(x;a)}=u[1](x;a)=u(x;a_1)+R(a_1), \tag{91}$$

which remains valid if we replace a by a_1, i.e.,

$$u(x;a_1)-2\frac{\mathrm{d}}{\mathrm{d}x}\frac{\psi_1'(x;a_1)}{\psi_1(x;a_1)}=u[1](x;a_1)=u(x;a_2)+R(a_2). \tag{92}$$

Hence, we easily obtain

$$u(x;a_1) = u[1](x;a_1) + 2\frac{\mathrm{d}}{\mathrm{d}x}\frac{\psi_1'(x;a_1)}{\psi_1(x;a_1)}. \tag{93}$$

2. Applying the Darboux transformation [Eq. (16)], once again on $u[1](x;a)$, gives

$$u[2](x;a) = u[1](x;a) - 2\frac{\mathrm{d}}{\mathrm{d}x}\frac{\psi'[1]_2(x;a)}{\psi[1]_2(x;a)}. \tag{94}$$

On the other hand, using the shape invariance condition leads to

$$u[2](x;a) = u(x;a_1) + R(a_1) - 2\frac{\mathrm{d}}{\mathrm{d}x}\frac{\psi'[1]_2(x;a)}{\psi[1]_2(x;a)}. \tag{95}$$

The result in the first step of the proof, [Eq. (93)] can be used to derive the following equation:

$$u[2](x;a) = u[1](x;a_1) + R(a_1) + 2\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{\psi_1'(x;a_1)}{\psi_1(x;a_1)} - \frac{\psi'[1]_2(x;a)}{\psi[1]_2(x;a)}\right\}. \tag{96}$$

If we now apply Eq. (88) from Lemma 5.1 for $s=1$, i.e.,

$$\psi[1]_2(x;a) = \psi_1(x;a_1), \tag{97}$$

we obtain the desired final expression which we wanted to prove, namely,

$$u[2](x;a) = u[1](x;a_1) + R(a_1). \tag{98}$$

For the sake of a more compact notation of the properties of the potential and wave functions, let us now call the property (88) *shape invariance for eigenfunctions* (or better *the two eigenfunctions involved are pairwise shape invariant*) and property (82) *shape invariance for the eigenvalues*. Note that the shape invariance of the wave functions follows from the shape invariance of the potentials. From the shape invariance of the eigenfunction we can, in turn, conclude that the next two pairs of Darboux transformations of the potential are pairwise shape invariant. One is led to the conjecture that the chain continues: from Lemma 5.2 one can show that the next pair of higher order Darboux transformations of eigenfunctions are also pairwise shape invariant, from which it follows that the next higher order pair of Darboux transformed potentials is also pairwise shape invariant. Indeed, we can prove the following theorem extending hereby the notion of shape invariance.

Theorem 5.3: *All neigbouring higher order Darboux transformed potentials and eigenfunctions are pairwise shape invariant. This is to say,*

$$u[k](x;a) = u[k-1]\ (x;a_1) + R(a_1)) \tag{99}$$

and

$$\psi[k]_{s+1}(x;a) = \psi[k-1]_s(x;a_1), \tag{100}$$

up to a multiplicative factor. In more detail, Eq. (99) *implies* Eq. (100) *which, in turn, implies*

$$u[k+1](x;a) = u[k](x;a_1) + R(a_1). \tag{101}$$

The proof proceeds via induction whose first step consists in Lemmas 5.2 and 5.1 or in Eqs. (88) and (98). We assume the hypothesis of the induction to be [Eq. (99) $\Rightarrow$ (100)]. This is sufficient since we start with the original shape invariance condition for potentials and the first step of induction is presented in Lemmas 5.1 and 5.2. We have to show that under this condition

$$\psi[k+1]_{s+1}(x;a) = \psi[k]_s(x;a_1) \tag{102}$$

holds, from which, in turn,

$$u[k+2](x;a) = u[k+1](x;a_1) + R(a_1) \tag{103}$$

follows.

1. We have

$$\left(-\frac{d^2}{dx^2} + u[k](x;a)\right)\psi[k]_s(x;a) = \lambda_s(a)\psi[k]_s(x;a), \quad s > k. \tag{104}$$

2. Since the above equation is valid for any a, it is also valid when a is replaced by $f(a)$. If we add $R(a_1)\psi[k]_s(x;a_1)$ on both sides and make use of the induction hypothesis we arrive, for $s > k$, at

$$\left(-\frac{d^2}{dx^2} + u[k+1](x;a)\right)\psi[k]_s(x;a_1) = (\lambda_s(a_1) + R(a_1))\psi[k]_s(x;a_1). \tag{105}$$

Equation (82) then tells us that

$$\psi[k]_s(x;a_1) = \psi[k+1]_{s+1}(x;a), \tag{106}$$

up to a multiplicative factor.

3. By definition we have

$$u[k+1](x;a) = u[k](x;a) - 2\frac{d}{dx}\left(\frac{\psi'[k]_{k+1}(x;a)}{\psi[k]_{k+1}(x;a)}\right), \tag{107}$$

for any a. Hence also

$$u[k+1](x;a_1) = u[k](x;a_1) - 2\frac{d}{dx}\left(\frac{\psi'[k]_{k+1}(x;a_1)}{\psi[k]_{k+1}(x;a_1)}\right). \tag{108}$$

Taking $u[k](x;a_1)$ from this equation and inserting the result in the induction hypothesis, one can easily show that

$$u[k+1](x;a) = u[k+1](x;a_1) + 2\frac{d}{dx}\left(\frac{\psi'[k]_{k+1}(x;a_1)}{\psi[k]_{k+1}(x;a_1)}\right) + R(a_1). \tag{109}$$

4. Again per definition we know that

$$u[k+2](x;a) = u[k+1](x;a) - 2\frac{d}{dx}\left(\frac{\psi'[k+1]_{k+2}(x;a)}{\psi[k+1]_{k+2}(x;a)}\right). \tag{110}$$

5. Combining the last two equations yields

$$u[k+2](x;a) = u[k+1](x;a_1) + R(a_1) + 2\frac{d}{dx}\left(\frac{\psi'[k]_{k+1}(x;a_1)}{\psi[k]_{k+1}(x;a_1)} - \frac{\psi'[k+1]_{k+2}(x;a)}{\psi[k+1]_{k+2}(x;a)}\right). \tag{111}$$

6. The last step consists in using the already established result [Eq. (106)] to obtain

$$u[k+2](x;a) = u[k+1](x;a_1) + R(a_1), \tag{112}$$

which completes the proof.

The shape invariance condition (more accurately, the shape invariance between u and $u[1]$) allows one to define a new Hamiltonian of the order s,

$$H_s^{\mathrm{SI}} \equiv -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + u(x;a_s) + \sum_{k=1}^{s} R(a_k). \tag{113}$$

Note that this definition makes no reference to higher order Darboux (or Crum) transformations. However, by virtue of the Theorem V.3 we can iterate

$$H_s^{\mathrm{SI}} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + u[1](x;a_{s-1}) + \sum_{k=1}^{s-1} R(a_k) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + u[2](x;a_{s-2}) + \sum_{k=1}^{s-2} R(a_k)$$
$$= \cdots = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + u[s](x;a) = H_s^D. \tag{114}$$

Hence, in view of the above and Theorem 3.1 we can state as a corollary.

Collary 5.4. *Under the condition of shape invariance all three transformations are equal, i.e.,*,

$$H_s^{\mathrm{SI}} = H_s^D = H_s^C. \tag{115}$$

VI. EXAMPLE

We will continue here with our example of the Morse potential, now, however, emphasizing the aspect of shape invariance around Lemmas 5.1 and 5.2 and Theorem 5.3. Indeed, the Morse potential is shape invariant. One defines the action of f as

$$f(A) \equiv A_1 = A - \frac{\alpha}{\sqrt{2}}, \tag{116}$$

in accordance with the notation in Eq. (40). R is identified with

$$R(A_1) = 2(A^2 - A_1^2). \tag{117}$$

Note that

$$\psi_1(x;A_1) = c_1(A_1)[\mathrm{sech}(\alpha x)]^{\sqrt{2}A_1/\alpha} = c_1(A_1)[\mathrm{sech}(\alpha x)]^{(\sqrt{2}A/\alpha)-1} = c\cosh(\alpha x)\psi_1(x;A) \tag{118}$$

immediately leads to

$$\psi[1]_2(x;A) = \psi_1(x;A_1), \tag{119}$$

which is valid up to a multiplicative constant. One also verifies that the equality below

$$\psi_2(x;A_1) = c_2(A_1)\sinh(\alpha x)\psi_1(x;A_1) \tag{120}$$

$$= c_2(A_1)\sinh(\alpha x)\cosh(\alpha x)\psi_1(x;A), \tag{121}$$

together with Eq. (44) has as a consequence the following identity (again up to a constant multiplicative value):

$$\psi[1]_3(x;A) = \psi_2(x;A_1). \tag{122}$$

Equations (119) and (122) are explicit examples of the result [Eq. (81) in Lemma 5.1. Regarding the eigenvalues, i.e., the property (82) in the same lemma, let us first note that another compact notation for Eq. (41) is

$$\lambda_n(A_1) = 2(A_1^2 - A_n^2), \tag{123}$$

which leaves us with the identity

$$\lambda_n(A_1) + R(A_1) = \lambda_{n+1}(A), \tag{124}$$

as it should be according to Lemma 5.1. Finally, we can also give explicit examples regarding Theorem 5.1. Due to the results from Sec. V, we can write

$$u[1](x;A_1) + R(A_1) = 2\left[A_1^2 - A_1\left(A_1 - \frac{\alpha}{\sqrt{2}}\right)\mathrm{sech}^2(\alpha x)\right] + 2(A^2 - A_1^2)$$
$$= 2[A^2 - A_1 A_2\,\mathrm{sech}^2(\alpha x)] = u[2](x;A). \tag{125}$$

This demonstrates in an explicit example the result [Eq. (99)] from Theorem 5.1. Last but not the least, one sees that Eq. (43) can be written as

$$\psi[1]_2(x;A_1) = c_2(A_1)\alpha\cosh(\alpha x)\psi_1(x;A_1), \tag{126}$$

which, according to Eq. (119) can be cast into the following form:

$$\psi[1]_2(x;A_1) = \frac{c_2(A_1)}{c}\alpha\cosh(\alpha x)\psi[1]_2(x;A) = \frac{c_2(A_1)c_2(A)}{c}\alpha^2\cosh^2(\alpha x)\psi_1(x;A) = \bar{c}\psi[2]_3(x;A), \tag{127}$$

with $\bar{c}$ a constant. To arrive at the last result we have used Eq. (57) from which one can also determine the constant $\bar{c}$ in terms of λ_3, $c_2(A)$, $c_2(A_1)$, and $c_3(A)$. Obviously, the above equation falls into the category of explicit examples of Eq. (100). Note that in none of the above examples we have used the actual lemmas or theorems to be exemplified (as it should be if an example carries some meaning).

VII. CONCLUSIONS

In the present work, we have clarified the relations between the Darboux and Crum transformations. We have shown that the latter can be reached iteratively by higher order Darboux transformations. This is valid for the potential as well as the eigenfunctions. If we subject the potential to the condition of shape invariance, another transform (not making use of Crum transformation for $n>1$) is possible [Eq. (113)]. We prove that this is also equivalent to the Crum transform. The main steps of this proof involved establishing Eqs. (99), (100), and (82). The first result, namely Eq. (99), is a generalization of the original shape invariance condition. Note that Eqs. (100) and (82) could be called shape invariance for the wave functions and eigenvalues. The results of the paper help to understand the relation between the different transformations of the Hamiltonian operator. Indeed, in view of our results, one could say that the Crum transformation which appears much more complex than the original Darboux result is essentially an iterative higher order Darboux transformation and therefore not more complex than the former.

APPENDIX: AN APPLICATION OF JACOBI THEOREM

The identity $W(W_n, W_{n-1,s}) = W_{ns}W_{n-1}$ has been used by Crum in the proof of his theorem. We have also made use of it several times in the present paper. It therefore makes sense to sketch a proof of the same.

Let us first establish some notations and definitions. Let $A=[a_{ij}]$ be an $n\times n$ matrix. The determinant of A will be denoted by $|A|$ as usual. We call the minor M_r, the determinant obtained by retaining from A the r lines $i_1, i_2, \ldots, i_r$ and the r columns $k_1, k_2, \ldots, k_r$. One defines the complement of the minor M_r as the determinant obtained from A by dropping the r lines $i_1, i_2, \ldots, i_r$ and the r columns $k_1, k_2, \ldots, k_r$. This complement will be denoted by M_r^c. One then defines $M^{(r)}$

$$M^{(r)} = (-1)^{i_1+i_2+\ldots+i_r+k_1+k_2+\ldots+k_r}M_r^c. \tag{A1}$$

Furthermore, let Δ be the matrix of the cofactors of A,

$$\Delta = \begin{vmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{vmatrix}, \tag{A2}$$

and M_r and M_r' the minors of $|A|$ and Δ, respectively.

Theorem (Jacobi): *With these notations, the theorem of Jacobi asserts that*

$$M_r' = |A|^{r-1} M^{(r)}. \tag{A3}$$

Before proceeding we make a small diversion to an example of the application of the above theorem starting with a Wronskian composed of ψ_1, ψ_2, ψ_3, and ψ_s, i.e.,

$$|A| \equiv W_{3,s} = \begin{vmatrix} \psi_1 & \psi_2 & \psi_3 & \psi_s \\ \psi_1' & \psi_2' & \psi_3' & \psi_s' \\ \psi_1'' & \psi_2'' & \psi_3'' & \psi_s'' \\ \psi_1''' & \psi_2''' & \psi_3''' & \psi_s''' \end{vmatrix}. \tag{A4}$$

The matrix of the cofactors is then given by

$$\Delta = \begin{vmatrix}
+\begin{vmatrix} \psi_2' & \psi_3' & \psi_s' \\ \psi_2'' & \psi_3'' & \psi_s'' \\ \psi_2''' & \psi_3''' & \psi_s''' \end{vmatrix} & -\begin{vmatrix} \psi_1' & \psi_3' & \psi_s' \\ \psi_1'' & \psi_3'' & \psi_s'' \\ \psi_1''' & \psi_3''' & \psi_s''' \end{vmatrix} & +\begin{vmatrix} \psi_1' & \psi_2' & \psi_s' \\ \psi_1'' & \psi_2'' & \psi_s'' \\ \psi_1''' & \psi_2''' & \psi_s''' \end{vmatrix} & -\begin{vmatrix} \psi_1' & \psi_2' & \psi_3' \\ \psi_1'' & \psi_2'' & \psi_3'' \\ \psi_1''' & \psi_2''' & \psi_3''' \end{vmatrix} \\
-\begin{vmatrix} \psi_2 & \psi_3 & \psi_s \\ \psi_2'' & \psi_3'' & \psi_s'' \\ \psi_2''' & \psi_3''' & \psi_s''' \end{vmatrix} & +\begin{vmatrix} \psi_1 & \psi_3 & \psi_s \\ \psi_1'' & \psi_3'' & \psi_s'' \\ \psi_1''' & \psi_3''' & \psi_s''' \end{vmatrix} & -\begin{vmatrix} \psi_1 & \psi_2 & \psi_s \\ \psi_1'' & \psi_2'' & \psi_s'' \\ \psi_1''' & \psi_2''' & \psi_s''' \end{vmatrix} & +\begin{vmatrix} \psi_1 & \psi_2 & \psi_3 \\ \psi_1'' & \psi_2'' & \psi_3'' \\ \psi_1''' & \psi_2''' & \psi_3''' \end{vmatrix} \\
+\begin{vmatrix} \psi_2 & \psi_3 & \psi_s \\ \psi_2' & \psi_3' & \psi_s' \\ \psi_2''' & \psi_3''' & \psi_s''' \end{vmatrix} & -\begin{vmatrix} \psi_1 & \psi_3 & \psi_s \\ \psi_1' & \psi_3' & \psi_s' \\ \psi_1''' & \psi_3''' & \psi_s''' \end{vmatrix} & +\begin{vmatrix} \psi_1 & \psi_2 & \psi_s \\ \psi_1' & \psi_2' & \psi_s' \\ \psi_1''' & \psi_2''' & \psi_s''' \end{vmatrix} & -\begin{vmatrix} \psi_1 & \psi_2 & \psi_3 \\ \psi_1' & \psi_2' & \psi_3' \\ \psi_1''' & \psi_2''' & \psi_3''' \end{vmatrix} \\
-\begin{vmatrix} \psi_2 & \psi_3 & \psi_s \\ \psi_2' & \psi_3' & \psi_s' \\ \psi_2'' & \psi_3'' & \psi_s'' \end{vmatrix} & +\begin{vmatrix} \psi_1 & \psi_3 & \psi_s \\ \psi_1' & \psi_3' & \psi_s' \\ \psi_1'' & \psi_3'' & \psi_s'' \end{vmatrix} & -\begin{vmatrix} \psi_1 & \psi_2 & \psi_s \\ \psi_1' & \psi_2' & \psi_s' \\ \psi_1'' & \psi_2'' & \psi_s'' \end{vmatrix} & +\begin{vmatrix} \psi_1 & \psi_2 & \psi_3 \\ \psi_1' & \psi_2' & \psi_3' \\ \psi_1'' & \psi_2'' & \psi_3'' \end{vmatrix}
\end{vmatrix}.$$

We choose as lines and columns: $(i_1,i_2)=(3,4)=(k_1,k_2)$. Applying the Jacobi theorem gives us

$$\begin{vmatrix}
+\begin{vmatrix} \psi_1 & \psi_2 & \psi_s \\ \psi_1' & \psi_2' & \psi_s' \\ \psi_1''' & \psi_2''' & \psi_s''' \end{vmatrix} & -\begin{vmatrix} \psi_1 & \psi_2 & \psi_3 \\ \psi_1' & \psi_2' & \psi_3' \\ \psi_1''' & \psi_2''' & \psi_3''' \end{vmatrix} \\
-\begin{vmatrix} \psi_1 & \psi_2 & \psi_s \\ \psi_1' & \psi_2' & \psi_s' \\ \psi_1'' & \psi_2'' & \psi_s'' \end{vmatrix} & +\begin{vmatrix} \psi_1 & \psi_2 & \psi_3 \\ \psi_1' & \psi_2' & \psi_3' \\ \psi_1'' & \psi_2'' & \psi_3'' \end{vmatrix}
\end{vmatrix} = W_{3,s} \begin{vmatrix} \psi_1 & \psi_2 \\ \psi_1' & \psi_2' \end{vmatrix}. \tag{A5}$$

Using Lemma 2.2 the left hand side takes the form

$$\begin{vmatrix} (\mathrm{d}/\mathrm{d}x)W_{2,s} & (\mathrm{d}/\mathrm{d}x)W_3 \\ W_{2,s} & W_3 \end{vmatrix}, \tag{A6}$$

such that we can write

$$\begin{vmatrix} (\mathrm{d}/\mathrm{d}x)W_{2,s} & -(\mathrm{d}/\mathrm{d}x)W_3 \\ -W_{2,s} & W_3 \end{vmatrix} = W_{3,s}W_2. \tag{A7}$$

Explicitly, this implies the following equality:

$$W_{3,s}W_2 = W_3\frac{\mathrm{d}}{\mathrm{d}x}W_{2,s} - W_{2,s}\frac{\mathrm{d}}{\mathrm{d}x}W_3 = W(W_3, W_{2,s}). \tag{A8}$$

The proof of the general case does not require any new procedure and follows essentially the steps outlined in the example. Let W_{ns} be the Wronskian of the $n+1$ functions $\psi_1, \ldots, \psi_n, \psi_s$, namely,

$$|A| = W_{n,s} = \begin{vmatrix} \psi_1 & \psi_2 & \ldots & \psi_n & \psi_s \\ \psi_1' & \psi_2' & \ldots & \psi_n' & \psi_s' \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \psi_1^{(n-1)} & \psi_2^{(n-1)} & \ldots & \psi_n^{(n-1)} & \psi_s^{(n-1)} \\ \psi_1^{(n)} & \psi_2^{(n)} & \ldots & \psi_n^{(n)} & \psi_s^{(n)} \end{vmatrix}, \tag{A9}$$

and Δ the matrix of the cofactors of $W_{n,s}$. We would like to apply the Jacobi theorem for the choice

$$(i_1, i_2) = (n, n+1) = (k_1, k_2), \tag{A10}$$

such that $r=2$. In this case we need

$$M_r' = \begin{vmatrix} +W_{n-1,s}' & -W_n' \\ -W_{n-1,s} & +W_n \end{vmatrix} = W(W_n, W_{n-1,s}), \tag{A11}$$

where we have used explicitly the result of Lemma 2.2. Clearly, we have,

$$M^{(r)} = W_{n-1}, \tag{A12}$$

such that the Jacobi theorem for the Wronskian A can be stated as

$$W(W_n, W_{n-1,s}) = W_{ns}W_{n-1}, \tag{A13}$$

which proves Lemma 2.3.

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