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Estimates of the Attraction Region for a Class of Nonlinear Time-Delay Systems

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Abstract

In this paper, we propose various estimates of the attraction region for a class of nonlinear time-delay systems of the form $\dot{x}(t) = Ax(t) + Bx(t - h(t)) + f(x(t), x(t - h(t)))$. The approach is constructive and makes use of a Lyapunov-Krasovskii functional associated to the linear part. Several illustrative examples (delayed logistic equation, stabilizing nonlinear oscillations by delayed output feedback, congestion control in high-performance networks and hereditary phenomena in physics) complete the presentation.

Keywords: nonlinear time-delay systems, attraction region, Lyapunov-Krasovskii functionals.

1 Introduction

It is well-known that computing estimates of the attraction region for a given nonlinear time-delay system is not a trivial task. The Lyapunov's second method is a powerful theoretical tool to solve this problem. Thus, if we are able to find an appropriate Lyapunov-Krasovskii functional or Lyapunov-Razumikhin function for the system under investigation then we can explicitly use it to construct an estimate of the attraction region, see, for instance, (Hale & Verduyn-Lunel 1993). However, the problem of finding a suitable Lyapunov-Krasovskii functional or Lyapunov-Razumikhin function for a given nonlinear time-delay system is far to be simple.

Confronted with such a problem in the general case, some effort has been devoted to find estimates of the attraction region for some classes of nonlinear time-delay systems which possess a linear part in their description. To the best of the authors' knowledge, only a few studies have addressed this problem. Thus, in (Kolmanovskii & Myshkis 1999) a bound based on the L_2 -stability is given. A method of computing estimates based on a comparison theorem and special vector Lyapunov functions is proposed in (Richard *et al.* 1997). Next, in (Verriest 2000) a stability analysis, going from linear to nonlinear, by means of Lyapunov-Krasovskii functionals which are known to work for the linear part is developed, and *delay-independent* estimates are derived. Finally, similar constructions and their use to motivate the fixed point analysis for functional differential equations can be found in (Burton 2006) [section 1.3].

In the finite dimensional case, it is well-known that for a system of the form $\dot{x} = Ax + f(x)$, where $\dot{x} = Ax$ is exponentially stable and $f(x)$ vanishes at the origin, an estimate of the attraction region is given by the set

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$\{x \in \mathbb{R}^n : v(x) = x^T P x < c\}$, where $c > 0$ is a constant depending on matrix P and function $f(x)$, see for instance (Khalil 1996). Here $P > 0$ is the solution of the Lyapunov equation: $A^T P + P A = -W$ for any chosen $W > 0$. This method is a consequence of the stability theorem on the first approximation which shows that the quadratic function $v(x) = x^T P x$ associated to the linear system is a Lyapunov function for the system in some neighborhood of the origin.

To the best of the authors' knowledge, there is no similar constructive method for delay systems of the form $\dot{x}(t) = Ax(t) + Bx(t-h) + f(x(t), x(t-h))$. This is, to provide estimates of the attraction region by using explicitly a quadratic Lyapunov-Krasovskii functional associated to the exponentially stable linear system $\dot{x}(t) = Ax(t) + Bx(t-h)$. A stability theorem on the first approximation for delay systems by means of Lyapunov-Krasovskii functionals can be found in (Halanay 1966) and (Lakshmikantham & Leela 1969). However, such a result does not provide an explicit procedure to determine an estimate of the attraction region for a given nonlinear time-delay system. A natural approach to address this problem is to use a particular Lyapunov-Krasovskii functional which could work for the linear system as a Lyapunov-Krasovskii functional candidate for the nonlinear system, as suggested by (Verriest 2000). However, such an approach is not along the lines of the finite dimensional case. Furthermore, as it is mentioned in (Verriest 2000), the resulting stability conditions are in general difficult to check when matrix B is not zero.

In our opinion, the difficulties arising in constructing quadratic Lyapunov-Krasovskii functionals associated to a given exponentially stable linear time-delay system seem to be the main reason for this lack. Some general expressions of quadratic functionals for linear time-delay systems have been proposed in the literature starting with the 60s: (Repin 1966), (Infante & Castelan 1978) and (Huang 1989). Whereas all these functionals are appropriate for stability analysis in the linear case, however, the forms of the corresponding Lyapunov derivatives as well as the construction itself complicate the study in the robust and nonlinear cases.

Recently, (Kharitonov & Zhabko 2003) proposed a new class of Lyapunov-Krasovskii functionals which overcomes such problems in the robust stability analysis case. Inspired by such a construction, we are able to compute estimates of the attraction region. More explicitly, we present a *constructive method of computing estimates of the attraction region* for some classes of nonlinear time-delay systems by making use of quadratic Lyapunov-Krasovskii functionals associated to the exponentially stable linear part. We believe that such an approach allows answering properly to the problem above.

The remaining part of the paper is organized as follows: In section 2, after some preliminaries we present the construction of Lyapunov-Krasovskii functional for linear time-delay systems according to (Kharitonov & Zhabko 2003). In section 3, the main results for the constant delay case are presented. The extension of the results to time-varying delay systems is presented in section 4. The examples in section 5 illustrate the results, and some concluding remarks end the paper. The notations are standard, and briefly presented in section 2.

2 Preliminaries

Consider the following class of time-delay systems:

$$\begin{cases} \dot{y}(t) &= Ay(t) + By(t-h(t)) + f(y(t), y(t-h(t))), \\ y(t) &= \psi(t), t \in \mathcal{E}_0, \end{cases} \quad (1)$$

where $h : \mathbb{R}_+ \mapsto \mathbb{R}_+$ denotes the delay, assumed to be a bounded function $0 \leq h(t) \leq H$ for all $t \geq 0$, and \mathcal{E}_0 is given by:

$$\mathcal{E}_0 = \{t \in \mathbb{R} : t = \eta - h(\eta) \leq 0, \quad \eta \geq 0\}.$$

The function $f(u, v)$ satisfies a Lipschitz condition in a certain neighborhood of the origin, and

$$\lim_{\|(u,v)\| \rightarrow 0} \frac{\|f(u, v)\|}{\|(u, v)\|} = 0. \quad (2)$$

From these assumptions, it follows that $f(0, 0) = 0$. In order to define a particular solution $y(t, \psi)$ of (1), an initial vector function $\psi(t)$, $t \in \mathcal{E}_0$ should be given. We assume that ψ belongs to the space of continuous vector functions mapping $\mathcal{E}_0 \subset [-H, 0]$ to \mathbb{R}^n equipped with the uniform norm $\|\psi\|_H = \sup_{\theta \in [-H, 0]} \|\psi(\theta)\|$ defined on the interval $[-H, 0]$. We denote by $y_t(\psi) = y(t+\theta, \psi)$, $\theta \in \mathcal{E}_0$, the translation of the solution $y(t, \psi)$ on $\mathcal{E}_0 \subset [-H, 0]$. Throughout this paper we will use the Euclidean norm for vectors and the induced matrix norm for matrices, both denoted by $\|\cdot\|$.

Definition 1 The trivial solution of (1) is stable if for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that $\|\psi\|_H < \delta$ implies $\|y(t, \psi)\| < \varepsilon$ for $t \geq 0$.

Definition 2 The trivial solution of (1) is asymptotic stable if it is stable and there exists $\delta_a > 0$ such that $\|\psi\|_H < \delta_a$ implies $y(t, \psi) \rightarrow 0$ as $t \rightarrow \infty$.

Definition 3 The trivial solution of (1) is exponentially stable if there exist constants $\delta_e > 0$, $\mu \geq 1$, and $\alpha > 0$ such that $\|\psi\|_H < \delta_e$ implies that $\|y(t, \psi)\| \leq \mu \|\psi\|_H e^{-\alpha t}$, $t \geq 0$.

From the stability theorem on the first approximation, we have that if the system

$$\dot{x}(t) = Ax(t) + Bx(t - h(t))$$

is exponentially stable, then the trivial solution of (1) is asymptotically stable, for *sufficiently small initial conditions*, see (Bellman & Cooke 1963), (Halanay 1966), (Lakshmikantham & Leela 1969), and (Hale & Verduyn-Lunel 1993).

Hence, according to the definition 2, there exists $\delta_a > 0$ (*sufficiently small*) such that $\|\psi\|_H < \delta_a$ implies that $y(t, \psi) \rightarrow 0$ as $t \rightarrow \infty$.

As mentioned in the Introduction, the main goal of this paper is to present a constructive procedure for computing estimates of the sphere $\|\psi\|_H < \delta_a$ in the space of the continuous functions mapping $[-H, 0]$ to \mathbb{R}^n such that $y(t, \psi) \rightarrow 0$ as $t \rightarrow \infty$, that is, to *compute an estimate of the attraction region whenever the first approximation is exponentially stable*.

We restrict our analysis to the following two particular classes of systems (1):

(a) *the constant delay case*

$$\dot{y}(t) = Ay(t) + By(t - h) + f(y(t), y(t - h))$$

(b) *the time-varying delay case*

$$\dot{y}(t) = Ay(t) + By(t - h(t)) + f(y(t), y(t - h(t))),$$

where $h(t) = h + \eta(t)$ with $0 \leq \eta(t) \leq \eta_0 \leq h$ and $\dot{\eta}(t) \leq \eta_1 < 1$.

As it was mentioned, the method developed in the sequel is along the lines of the finite-dimensional case for computing estimates of the attraction region when the first approximation is exponentially stable (Halanay 1966) and (Khalil 1996). Thus, we first construct a quadratic Lyapunov-Krasovskii functional *associated* to the exponential stable system

$$\begin{cases} \dot{x}(t) &= Ax(t) + Bx(t - h) \\ x(t) &= \varphi(t), t \in [-h, 0]. \end{cases} \quad (3)$$

Next, we show that such a functional is a Lyapunov-Krasovskii functional for the nonlinear system, and we compute an estimate of the attraction region by using explicitly the functional.

In the rest of this section we present the construction procedure of quadratic Lyapunov-Krasovskii functionals *associated* to (3) according to (Kharitonov & Zhabko 2003).

Assume that the system (3) is exponentially stable, i.e., for any $\varphi \in \mathcal{C}([-h, 0], \mathbb{R}^n)$ there exist some constants $\mu \geq 1$ and $\alpha > 0$ such that

$$\|x(t, \varphi)\| \leq \mu \|\varphi\|_h e^{-\alpha t}, \quad t \geq 0.$$

Next, consider the following functional

$$w(\varphi) = \varphi^T(0)W_0\varphi(t) + \varphi^T(-h)W_1\varphi(-h) + \int_{-h}^0 \varphi^T(\theta)W_2\varphi(\theta)d\theta,$$

where the symmetric and positive definite matrices W_j , $j = 0, 1, 2$ are appropriately selected, and $\varphi \in \mathcal{C}([-h, 0], \mathbb{R}^n)$ is arbitrary. If the system (3) is exponentially stable, then there exists a unique quadratic functional $v : \mathcal{C}([-h, 0], \mathbb{R}^n) \mapsto \mathbb{R}$ such that the mapping $t \mapsto v(x_t(\varphi))$ is differentiable for $t \geq 0$ and

$$\frac{dv(x_t(\varphi))}{dt} = -w(x_t(\varphi)), \quad t \geq 0,$$

for all solutions $x(t, \varphi)$ of (3). The functional $v(\cdot)$ is called the Lyapunov-Krasovskii functional *associated* to the system (3) (Kharitonov & Zhabko 2003), and it is of the form

$$v(\varphi) = \varphi^T(0)U(0)\varphi(0) - 2\varphi^T(0) \int_{-h}^0 U(-h-\theta)B\varphi(\theta)d\theta \quad (4)$$

$$+ \int_{-h}^0 \int_{-h}^0 \varphi^T(\theta_1)B^T U(\theta_1 - \theta_2)B\varphi(\theta_2)d\theta_1 d\theta_2 + \int_{-h}^0 \varphi^T(\theta) (W_1 + (h + \theta)W_2) \varphi(\theta)d\theta,$$

where the matrix function $U(\cdot)$ is defined as

$$U(\tau) = \int_0^\infty K^T(t)WK(t + \tau)dt, \tau \in [-h, h], \quad (5)$$

where $W = W_0 + W_1 + hW_2$, and $K(t)$ is the unique matrix function which satisfies

$$\dot{K}(t) = AK(t) + BK(t - h), t > 0$$

with the initial condition $K(t) = 0$ for all $t < 0$, and $K(0) = I$, see (Bellman & Cooke 1963) and (Hale & Verduyn-Lunel 1993).

Note that by the exponential stability of (3) the matrix $U(\cdot)$ is well-defined. The matrix function $U(\cdot)$ satisfies the following second-order ordinary differential equation, see (Huang 1989) and (Kharitonov & Zhabko 2003):

$$\ddot{U}(\tau) = \dot{U}(\tau)A - A^T\dot{U}(\tau) + A^T U(\tau)A - B^T U(\tau)B \quad (6)$$

with the additional conditions

$$\dot{U}(0) = U(0)A + U^T(h)B, \quad (7)$$

$$-W = \dot{U}(0) + \left(\dot{U}(0)\right)^T. \quad (8)$$

Remark 1 The matrix function $U(\cdot)$ is known as a delay Lyapunov matrix associated to the system (3). When the system (3) is exponentially stable, then the matrix function (5) is the unique solution of (6) satisfying (7) and (8), see, for instance, (Kharitonov & Plischke 2006). A piece-wise linear approximation of $U(\cdot)$ can be computed from equations (6)-(8) (Kharitonov & Garcia-Lozano 2004). In the scalar case, it is possible to obtain an explicit solution of the equations (6)-(8), as shown by (Melchor-Aguilar 2004).

Proposition 1 Let the system (3) be exponentially stable. Given any positive definite matrices $W_j, j = 0, 1, 2$, the functional (4) satisfies

1. $\alpha_1 \|\varphi(0)\|^2 \leq v(\varphi) \leq \alpha_2 \|\varphi\|_h^2$, for some constants $\alpha_1 > 0$ and $\alpha_2 > 0$,
2. $\frac{d}{dt}v(x_t(\varphi)) \leq -\beta \|x(t, \varphi)\|^2$, for some constant $\beta > 0$.

Proof. First observe that given any $W_j > 0, j = 0, 1, 2$, the exponential stability of (3) implies that functional $v(\varphi)$ is well defined by (4) and (5). Moreover, the functional $w(\varphi)$ is well defined and is related with $v(\varphi)$ by

$$\frac{dv(x_t(\varphi))}{dt} = -w(x_t(\varphi)), t \geq 0.$$

Then

$$\frac{dv(x_t(\varphi))}{dt} \leq -\beta \|x(t, \varphi)\|^2, t \geq 0,$$

where $\beta = \lambda_{\min}(W_0)$. In order to prove the lower bound for $v(\varphi)$, consider the functional

$$v_\epsilon(\varphi) = v(\varphi) - \epsilon\varphi^T(0)\varphi(0).$$

We have

$$\frac{dv_\epsilon(x_t(\varphi))}{dt} = -w(x_t(\varphi)) - 2\epsilon x^T(t, \varphi) (Ax(t, \varphi) + Bx(t - h, \varphi)), t \geq 0.$$

Considering the inequality

$$-2\epsilon x^T(t, \varphi) (Ax(t, \varphi) + Bx(t-h, \varphi)) \leq 2\epsilon \|A\| \|x(t, \varphi)\|^2 + \epsilon \|B\| (\|x(t, \varphi)\|^2 + \|x(t-h, \varphi)\|^2)$$

we arrive at

$$\frac{dv_\epsilon(x_t(\varphi))}{dt} \leq -w_\epsilon(x_t(\varphi)), \quad t \geq 0, \quad (9)$$

where

$$w_\epsilon(\varphi) = [\lambda_{\min}(W_0) - \epsilon(2\|A\| + \|B\|)] \|\varphi(0)\|^2 + [\lambda_{\min}(W_1) - \epsilon\|B\|] \|\varphi(-h)\|^2.$$

Choosing $\epsilon > 0$ such that

$$\epsilon < \min \left\{ \frac{\lambda_{\min}(W_0)}{2\|A\| + \|B\|}, \frac{\lambda_{\min}(W_1)}{\|B\|} \right\}, \quad (10)$$

we have $w_\epsilon(\varphi) \geq 0$. Integrating from 0 to ∞ the inequality (9) we get

$$v_\epsilon(\varphi) \geq \int_0^\infty w_\epsilon(x_t(\varphi)) dt \geq 0,$$

which implies that

$$v(\varphi) \geq \alpha_1 \|\varphi(0)\|^2,$$

for $\alpha_1 = \epsilon$. Let

$$u_0 = \max_{\tau \in [0, h]} \|U(\tau)\|.$$

The following inequalities can be easily checked:

$$\begin{aligned} \varphi^T(0)U(0)\varphi(0) &\leq u_0 \|\varphi(0)\|^2, \\ -2\varphi^T(0) \int_{-h}^0 U(-h-\theta)B\varphi(\theta)d\theta &\leq u_0 \|B\| \left(h \|\varphi(0)\|^2 + \int_{-h}^0 \|\varphi(\theta)\|^2 d\theta \right), \\ \int_{-h}^0 \int_{-h}^0 \varphi^T(\theta_1)B^T U(\theta_1 - \theta_2)B\varphi(\theta_2)d\theta_1 d\theta_2 &\leq u_0 h \|B\|^2 \int_{-h}^0 \|\varphi(\theta)\|^2 d\theta, \\ \int_{-h}^0 \varphi^T(\theta) (W_1 + (h+\theta)W_2) \varphi(\theta)d\theta &\leq \|W_1 + hW_2\| \int_{-h}^0 \|\varphi(\theta)\|^2 d\theta. \end{aligned}$$

Then, the following inequality holds:

$$v(\varphi) \leq \kappa \left(\|\varphi(0)\|^2 + \int_{-h}^0 \|\varphi(\theta)\|^2 d\theta \right), \quad (11)$$

where

$$\kappa = \max \{u_0(1+h\|B\|), u_0\|B\|(1+h\|B\|) + \|W_1 + hW_2\|\}.$$

Hence we get

$$v(\varphi) \leq \alpha_2 \|\varphi\|_h^2, \quad \text{for } \alpha_2 \geq \kappa(1+h).$$

□

3 Constant Delay Case

In this section, we show that an estimate of the attraction region for the trivial solution of

$$\begin{cases} \dot{y}(t) &= Ay(t) + By(t-h) + f(y(t), y(t-h)) \\ y(t) &= \psi(t), \quad t \in [-h, 0] \end{cases} \quad (12)$$

can be computed when the linear system (3) is exponentially stable.

From (2) it follows that for any $\gamma > 0$ there exists $\delta = \delta(\gamma) > 0$ such that

$$\|f(y(t), y(t-h))\| < \gamma \|(y(t), y(t-h))\| \quad \text{if } \|(y(t), y(t-h))\| < \delta.$$

Theorem 1 Let system (3) be exponentially stable. For

$$0 < \gamma < \min \left\{ \frac{\lambda_{\min}(W_0)}{u_0(2 + \|B\|h)}, \frac{\lambda_{\min}(W_1)}{u_0(1 + \|B\|h)}, \frac{\lambda_{\min}(W_2)}{\|B\|u_0} \right\}, \quad (13)$$

the set

$$\mathcal{U} = \left\{ \psi \in \mathcal{C}([-h, 0], \mathbb{R}^n) : v(\psi) < \frac{\alpha_1 \delta^2}{4} \text{ and } \|\psi\|_h < \frac{\delta}{2} \right\} \quad (14)$$

is an estimate of the attraction region for the trivial solution of (12).

Proof. Let system (3) be exponentially stable. Then, given any $W_j > 0$, $j = 0, 1, 2$, there exists a unique function $U(\cdot)$ satisfying equations (6)-(8), and therefore a unique functional (4) associated to (3). We will show that (4) is a Lyapunov-Krasovskii functional candidate for (12).

The derivative of (4), along solutions of (12), is:

$$\begin{aligned} \frac{dv(y_t(\psi))}{dt} &= -w(y_t(\psi)) \\ &+ 2f^T(y(t, \psi), y(t-h, \psi)) \left(U(0)y(t, \psi) - \int_{-h}^0 U(-h-\theta)By(t+\theta, \psi)d\theta \right). \end{aligned} \quad (15)$$

For any trajectory $y_t(\psi) \in \mathcal{U}$, we have $\|(y(t, \psi), y(t-h, \psi))\| < \delta$ which implies that

$$\|f(y(t, \psi), y(t-h, \psi))\| < \gamma \|(y(t, \psi), y(t-h, \psi))\|.$$

The following inequality holds:

$$\left\| U(0)y(t, \psi) - \int_{-h}^0 U(-h-\theta)By(t+\theta, \psi)d\theta \right\| \leq u_0 \left(\|y(t, \psi)\| + \|B\| \int_{-h}^0 \|y(t+\theta, \psi)\| d\theta \right). \quad (16)$$

It follows that

$$\begin{aligned} &2f^T(y(t, \psi), y(t-h, \psi)) \left(U(0)y(t, \psi) - \int_{-h}^0 U(-h-\theta)By(t+\theta, \psi)d\theta \right) \\ &\leq \gamma u_0 \left[(2 + \|B\|h) \|y(t, \psi)\|^2 + (1 + \|B\|h) \|y(t-h, \psi)\|^2 + \|B\| \int_{-h}^0 \|y(t+\theta, \psi)\|^2 d\theta \right]. \end{aligned}$$

Considering this inequality in (15) we arrive to the following inequality:

$$\begin{aligned} \frac{dv(y_t, \psi)}{dt} &\leq -[\lambda_{\min}(W_0) - \gamma u_0(2 + \|B\|h)] \|y(t, \psi)\|^2 \\ &- [\lambda_{\min}(W_1) - \gamma u_0(1 + \|B\|h)] \|y(t-h, \psi)\|^2 \\ &- [\lambda_{\min}(W_2) - \|B\|u_0\gamma] \int_{-h}^0 \|y(t+\theta, \psi)\|^2 d\theta. \end{aligned} \quad (17)$$

Choosing $\gamma > 0$ satisfying inequality (13), we have $\frac{d}{dt}v(y_t(\psi)) < 0$, $t \geq 0$, for trajectories of (12) inside \mathcal{U} , implying thus the asymptotic stability of the trivial solution of (12).

Now we show that the set \mathcal{U} is a positively invariant set with respect to (12). For any initial function $\psi \in \mathcal{U}$, we have $\|\psi\|_h < \frac{\delta}{2}$, and it follows that $\frac{d}{dt}v(y_t(\psi))|_{t=0} < 0$. Then, there exists $t_0 > 0$, sufficiently small, such that

$$v(y_t(\psi)) < v(\psi) < \frac{\alpha_1 \delta^2}{4}, \quad t \in (0, t_0).$$

From the lower bound of functional $v(\cdot)$ we obtain $\|y(t, \psi)\| < \frac{\delta}{2}$, $t \in (0, t_0)$. Since $\|y(t, \psi)\| = \|\psi(t)\| < \frac{\delta}{2}$, for $t \in [t_0 - h, 0]$, it follows that

$$\|y_t(\psi)\|_h < \frac{\delta}{2}, \quad t \in [0, t_0].$$

Hence $y_t(\psi) \in \mathcal{U}$, $t \in [0, t_0)$. Now we assume that $y_t(\psi) \in \mathcal{U}$ is not satisfied for all $t \geq 0$, and let $t_1 \geq t_0$ be the upper bound of the set of values t for which it still holds. We have

$$\|y(t_1, \psi)\| = \frac{\delta}{2}. \quad (18)$$

Since $t \mapsto v(y_t(\psi))$ is a continuous function of t , we have

$$v(y_{t_1}(\psi)) < v(\psi), \quad t \in (0, t_1)$$

and

$$\alpha_1 \|y(t_1, \psi)\| \leq v(y_{t_1}(\psi)) \leq v(\psi) < \alpha_1 \frac{\delta^2}{4}.$$

So, we have $\|y(t_1, \psi)\| < \frac{\delta}{2}$, which contradicts (18). The existence of t_1 is contradictory and it follows that $y_t(\psi) \in \mathcal{U}$ for all $t \geq 0$.

Therefore, the set \mathcal{U} is a positively invariant set with respect to (12), and (4) is a Lyapunov-Krasovskii functional on \mathcal{U} , which implies that the set \mathcal{U} is an estimate of the attraction region. \square

Corollary 1 *Let system (3) be exponentially stable. For any solution $y(t, \psi)$ of (12) such that $\psi \in \mathcal{U}$, the following exponential estimate holds:*

$$\|y(t, \psi)\| \leq \sqrt{\frac{\alpha_2}{\alpha_1}} \|\psi\|_h e^{-\frac{\beta}{2\kappa}t}, \quad t \geq 0, \quad (19)$$

where

$$\beta = \min \{ \lambda_{\min}(W_0) - \gamma u_0 (2 + \|B\| h), \lambda_{\min}(W_2) - \|B\| u_0 \gamma \}.$$

Proof. For any initial function $\psi \in \mathcal{U}$ we have $y_t(\psi) \in \mathcal{U}$, for all $t \geq 0$, since \mathcal{U} is a positively invariant set with respect to (12). From inequality (17) we obtain

$$\frac{d}{dt} v(y_t(\psi)) \leq -\beta \left(\|y(t, \psi)\|^2 + \int_{-h}^0 \|y(t + \theta, \psi)\|^2 d\theta \right), \quad t \geq 0,$$

where

$$\beta = \min \{ \lambda_{\min}(W_0) - \gamma u_0 (2 + \|B\| h), \lambda_{\min}(W_2) - \|B\| u_0 \gamma \}.$$

From (11) we have

$$v(y_t(\psi)) \leq \kappa \left(\|y(t, \psi)\|^2 + \int_{-h}^0 \|y(t + \theta, \psi)\|^2 d\theta \right), \quad t \geq 0.$$

It follows that

$$\frac{d}{dt} v(y_t(\psi)) \leq -\frac{\beta}{\kappa} v(y_t(\psi)), \quad t \geq 0.$$

Integrating this inequality from 0 to t we get

$$\ln v(y_t(\psi)) - \ln v(\psi) \leq -\frac{\beta}{\kappa} t,$$

from which it follows that

$$\alpha_1 \|y(t, \psi)\|^2 \leq v(y_t(\psi)) \leq v(\psi) e^{-\frac{\beta}{\kappa}t} \leq \alpha_2 \|\psi\|_h^2 e^{-\frac{\beta}{\kappa}t}, \quad t \geq 0,$$

and

$$\|y(t, \psi)\| \leq \sqrt{\frac{\alpha_2}{\alpha_1}} \|\psi\|_h e^{-\frac{\beta}{2\kappa}t}, \quad t \geq 0. \quad \square$$

Hence, according to the definition 3, the trivial solution of (12) is exponentially stable. Note that, by combining \mathcal{U} with the upper bound for the functional $v(\cdot)$ we can obtain the following more conservative, yet computationally more tractable estimate of the attraction region:

$$\mathcal{V} = \left\{ \psi \in \mathcal{C}([-h, 0], \mathbb{R}^n) : \|\psi\|_h < \sqrt{\frac{\alpha_1 \delta}{\alpha_2}} \frac{\delta}{2} \right\} \subseteq \mathcal{U}. \quad (20)$$

As in the case of systems without delay, estimating the region of attraction by the means of the set \mathcal{U} is simple but usually conservative. Clearly, the estimates of the attraction region (14) and (20) depend on the choice of the matrices W_j , $j = 0, 1, 2$. These matrices can be used as free parameters in order to optimize such estimates.

4 Time-varying delay case

In this section, we present the extension of the results for the time-varying delay case. We show that an estimate of the attraction region for the trivial solution of the system

$$\begin{cases} \dot{y}(t) &= Ay(t) + By(t - h(t)) + f(y(t), y(t - h(t))), \\ y(t) &= \psi(t), t \in [-2h, 0], \end{cases} \quad (21)$$

where $h(t) = h + \eta(t)$ with

$$0 \leq \eta(t) \leq \eta_0 \leq h \text{ and } \dot{\eta}(t) \leq \eta_1 < 1 \quad (22)$$

can be computed when the linear system

$$\begin{cases} \dot{y}(t) &= Ay(t) + By(t - h(t)) \\ y(t) &= \psi(t), t \in [-2h, 0] \end{cases} \quad (23)$$

is exponentially stable. The stability of (23) can be ensured from a robust stability approach for uncertain delay by assuming the stability of the nominal system (3), see (Kharitonov & Niculescu 2003).

Since for $t \geq 2h$

$$y(t - h(t)) - y(t - h) = - \int_{-h(t)}^{-h} \dot{y}(t + \theta) d\theta = - \int_{-h(t)}^{-h} [Ay(t + \theta) + By(t + \theta - h(t + \theta))], \quad (24)$$

we can write (23) as

$$\begin{aligned} \dot{y}(t) &= Ay(t) + By(t - h) - B \int_{-h(t)}^{-h} [Ay(t + \theta) + By(t + \theta - h(t + \theta))] \\ y(t) &= \varphi(t), t \in [-2h, 2h], \end{aligned} \quad (25)$$

where

$$\varphi(t) = \begin{cases} \psi(t), t \in [-2h, 0] \\ y(t, \psi), t \in (0, 2h] \end{cases}.$$

The method of transforming (23) to (25) is known as a model transformation (Gu & Niculescu 2000; Gu & Niculescu 2001; Kharitonov & Melchor-Aguilar 2000; Kharitonov & Melchor-Aguilar 2002). Clearly, every solution of (23) is a solution of (25), and therefore the stability of (25) implies the stability of (23). It is well known that such model transformation introduces some additional dynamics, which are not present in the original system (23), and a significant conservatism on the stability conditions may be obtained from the transformed system (25), see (Gu & Niculescu 2000; Gu & Niculescu 2001; Kharitonov & Melchor-Aguilar 2000; Kharitonov & Melchor-Aguilar 2002). As we will see later, the model transformation will also introduce a certain conservatism in our estimates of the attraction region.

Now, we construct an appropriate Lyapunov functional associated to (3) for the stability of (25). Assume that system (3) is exponentially stable. Selecting positive definite matrices $W_j, j = 0, 1, 2$, consider the functional

$$w(\varphi) = \varphi^T(0)W_0\varphi(0) + \varphi^T(-h)W_1\varphi(-h) + \int_{-4h}^0 \varphi^T(\theta)W_2\varphi(\theta)d\theta,$$

where $\varphi \in \mathcal{C}([-4h, 0], \mathbb{R}^n)$ is arbitrary. If system (3) is exponentially stable, then there exists a unique quadratic functional $v : \mathcal{C}([-h, 0], \mathbb{R}^n) \mapsto \mathbb{R}$ associated to (3) such that $t \mapsto v(\tilde{x}_t(\varphi))$ is differentiable for $t \geq 0$, and

$$\frac{dv(\tilde{x}_t(\varphi))}{dt} = -w(x_t(\varphi)), t \geq 0,$$

for all solutions $x(t, \varphi)$ of (3), where $\tilde{x}_t(\varphi) := x(t + \theta, \varphi), \theta \in [-h, 0]$ and $x_t(\varphi) := x(t + \theta, \varphi), \theta \in [-4h, 0]$.

The functional is of the form

$$\begin{aligned} v(\varphi) &= \varphi^T(0)U(0)\varphi(0) - 2\varphi^T(0) \int_{-h}^0 U(-h - \theta)B\varphi(\theta)d\theta \\ &+ \int_{-h}^0 \int_{-h}^0 \varphi^T(\theta_1)B^T U(\theta_1 - \theta_2)B\varphi(\theta_2)d\theta_1d\theta_2 + \int_{-h}^0 \varphi^T(\theta) (W_1 + (4h + \theta)W_2) \varphi(\theta)d\theta, \end{aligned} \quad (26)$$

where the matrix function $U(\cdot)$ is defined by (5) with $W = W_0 + W_1 + 4hW_2$, and satisfies the second order differential equation (6) with additional conditions (7) and (8). The functional (26) satisfies the conditions of proposition 1. Indeed, we have $\alpha_1 \|\varphi(0)\|^2 \leq v(\varphi)$ for

$$\alpha_1 < \left\{ \frac{\lambda_{\min}(W_0)}{2\|A\| + \|B\|}, \frac{\lambda_{\min}(W_1)}{\|B\|} \right\}. \quad (27)$$

The functional (26) satisfies

$$v(\varphi) \leq \kappa \left(\|\varphi(0)\|^2 + \int_{-h}^0 \|\varphi(\theta)\|^2 d\theta \right), \quad (28)$$

where

$$\kappa = \max \{ u_0 (1 + h \|B\|), u_0 \|B\| (1 + h \|B\|) + \|W_1 + 4hW_2\| \}.$$

Then

$$v(\varphi) \leq \alpha_2 \|\varphi\|_h^2, \text{ for } \alpha_2 \geq \kappa(1 + h).$$

Proposition 2 *System (23) is exponentially stable for a continuous time-varying function $\eta(t)$ satisfying (22), if system (3) is exponentially stable and for any given matrices $W_j > 0, j = 0, 1, 2$, there exist constants $k_j > 0, j = 1, 2, 3, 4$ such that the following inequalities hold:*

$$\begin{aligned} \lambda_{\min}(W_0) &> \eta_0 u_0 \|B\| (k_1^{-1} \|A\| + k_3^{-1} \|B\|) \\ \lambda_{\min}(W_2) &> \eta_0 u_0 \|B\|^2 (k_2^{-1} \|A\| + k_4^{-1} \|B\|) \\ \lambda_{\min}(W_2) &> \|A\| \|B\| u_0 (k_1 + k_2 \|B\| h) \\ \lambda_{\min}(W_2) &> (1 - \eta_1)^{-1} \|B\|^2 u_0 (k_3 + k_4 \|B\| h) \end{aligned} \quad (29)$$

Proof. Let the system (3) be exponentially stable. Then, given any matrices $W_j > 0, j = 0, 1, 2$, there exists a unique matrix function $U(\cdot)$ satisfying equations (6)-(8), and therefore a unique functional (26) associated to (3). For the solution $y(t, \varphi)$ of (25), consider $\tilde{y}_t(\varphi) := y(t + \theta, \varphi), \theta \in [-h, 0], t \geq 2h$ and $y_t(\varphi) := y(t + \theta, \varphi), \theta \in [-4h, 0], t \geq 2h$.

The derivative of (26), along solutions of (25), for $t \geq 2h$, is

$$\begin{aligned} \frac{dv(\tilde{y}_t(\varphi))}{dt} &= -w(y_t(\varphi)) - 2 \left(\int_{-h(t)}^{-h} [BAy(t + \theta, \varphi) + B^2y(t + \theta - h(t + \theta), \varphi)] d\theta \right)^T \times \\ &\times \left(U(0)y(t, \varphi) - \int_{-h}^0 U(-h - \theta)By(t + \theta, \varphi)d\theta \right). \end{aligned}$$

Considering inequality (16) and

$$\begin{aligned} &\left\| \int_{-h(t)}^{-h} [BAy(t + \theta, \varphi) + B^2y(t + \theta - h(t + \theta), \varphi)] d\theta \right\| \\ &\leq \|B\| \int_{-h(t)}^{-h} (\|A\| \|y(t + \theta, \varphi)\| + \|B\| \|y(t + \theta - h(t + \theta), \varphi)\|) d\theta \end{aligned}$$

we arrive to the following inequality:

$$\begin{aligned} \frac{dv(\tilde{y}_t(\varphi))}{dt} &\leq -w(y_t(\varphi)) + 2u_0 \|B\| \left(\|y(t, \varphi)\| + \|B\| \int_{-h}^0 \|y(t + \theta, \varphi)\| d\theta \right) \times \\ &\times \left(\int_{-h(t)}^{-h} (\|A\| \|y(t + \theta, \varphi)\| + \|B\| \|y(t + \theta - h(t + \theta), \varphi)\|) d\theta \right). \end{aligned}$$

The following inequalities hold:

$$\begin{aligned}
& 2 \|y(t, \varphi)\| \int_{-h(t)}^{-h} \|y(t + \theta, \varphi)\| d\theta \leq k_1^{-1} \eta_0 \|y(t, \varphi)\|^2 + k_1 \int_{-h(t)}^{-h} \|y(t + \theta, \varphi)\|^2 d\theta, \\
& 2 \left(\int_{-h(t)}^{-h} \|y(t + \theta, \varphi)\| d\theta \right) \left(\int_{-h}^0 \|y(t + \theta, \varphi)\| d\theta \right) \\
\leq & k_2^{-1} \eta_0 \int_{-h}^0 \|y(t + \theta, \varphi)\|^2 d\theta + k_2 h \int_{-h(t)}^{-h} \|y(t + \theta, \varphi)\|^2 d\theta, \\
& 2 \|y(t, \varphi)\| \int_{-h(t)}^{-h} \|y(t + \theta - h(t + \theta), \varphi)\| d\theta \\
\leq & k_3^{-1} \eta_0 \|y(t, \varphi)\|^2 + k_3 \int_{-h(t)}^{-h} \|y(t + \theta - h(t + \theta), \varphi)\|^2 d\theta, \\
& 2 \left(\int_{-h(t)}^{-h} \|y(t + \theta - h(t + \theta), \varphi)\| d\theta \right) \left(\int_{-h}^0 \|y(t + \theta, \varphi)\| d\theta \right) \\
\leq & k_4^{-1} \eta_0 \int_{-h}^0 \|y(t + \theta, \varphi)\|^2 d\theta + k_4 h \int_{-h(t)}^{-h} \|y(t + \theta - h(t + \theta), \varphi)\|^2 d\theta,
\end{aligned}$$

where $k_j, j = 1, 2, 3, 4$ are any free positive constants.

Observing that

$$\int_{-h(t)}^{-h} \|y(t + \theta, \varphi)\|^2 d\theta \leq \int_{-2h}^{-h} \|y(t + \theta, \varphi)\|^2 d\theta, \quad (30)$$

$$\begin{aligned}
\int_{-h(t)}^{-h} \|y(t + \theta - h(t + \theta), \varphi)\|^2 d\theta &= \int_{-h(t)-h(t-h)}^{-h-h(t-h)} \|y(t + \xi, \varphi)\| (1 - \dot{\eta}(t + \theta))^{-1} d\xi \\
&\leq (1 - \eta_1)^{-1} \int_{-4h}^{-2h} \|y(t + \xi, \varphi)\| d\xi,
\end{aligned} \quad (31)$$

we obtain

$$\begin{aligned}
\frac{dv(\tilde{y}_t(\varphi))}{dt} &\leq -w(y_t(\varphi)) + \eta_0 u_0 \|B\| (k_1^{-1} \|A\| + k_3^{-1} \|B\|) \|y(t, \varphi)\|^2 \\
&+ \eta_0 u_0 \|B\|^2 (k_2^{-1} \|A\| + k_4^{-1} \|B\|) \int_{-h}^0 \|y(t + \theta, \varphi)\|^2 d\theta \\
&+ \|A\| \|B\| u_0 (k_1 + k_2 \|B\| h) \int_{-2h}^{-h} \|y(t + \theta, \varphi)\|^2 d\theta \\
&+ (1 - \eta_1)^{-1} \|B\|^2 u_0 (k_3 + k_4 \|B\| h) \int_{-4h}^{-2h} \|y(t + \theta, \varphi)\|^2 d\theta.
\end{aligned}$$

If inequalities (29) are satisfied then $\frac{d}{dt}v(\tilde{y}_t(\varphi)) < 0$, for $t \geq 2h$, which implies the exponential stability of (23). \square

Now we rewrite (21) using formula (24) as

$$\begin{cases} \dot{y}(t) &= Ay(t) + By(t-h) + Bz(t) + f(y(t), y(t-h) + z(t)), \\ y(t) &= \varphi(t), t \in [-2h, 2h], \end{cases} \quad (32)$$

where

$$z(t) = - \int_{-h(t)}^{-h} Ay(t + \theta) + By(t + \theta - h(t + \theta)) + f(y(t + \theta), y(t + \theta - h(t + \theta))) d\theta$$

and

$$\varphi(t) = \begin{cases} \psi(t), & t \in [-2h, 0] \\ y(t, \psi), & t \in (0, 2h] \end{cases}. \quad (33)$$

The stability of the trivial solution of (32) implies that of the trivial solution of (21).

From (2) it follows that for any $\gamma > 0$ there exists $\delta = \delta(\gamma) > 0$ such that if $\|(y(t + \theta_1), y(t + \theta_2))\| < \delta$, for $\theta_1, \theta_2 \in [-4h, 0], t \geq 2h$, then

$$\|f(y(t + \theta_1), y(t + \theta_2))\| < \gamma \|(y(t + \theta_1), y(t + \theta_2))\|.$$

Theorem 2 *Let the system (3) be exponentially stable and assume that for any given matrices $W_j > 0, j = 0, 1, 2$, there exist constants $k_j > 0, j = 1, 2, 3, 4$ such that inequalities (29) hold. For*

$$0 < \gamma < \min_{j=0,1,2,3} \left\{ \gamma_{rj}, \frac{\lambda_{\min}(W_1)}{b} \right\}, \quad (34)$$

where $b = u_0(1 + \|B\|h)$, and $\gamma_{rj}, j = 0, 1, 2, 3$ are the corresponding positive real roots of the equations

$$a_j \gamma^2 + b_j \gamma + c_j = d_j, \quad j = 0, 1, 2, 3 \quad (35)$$

with $d_0 = \lambda_{\min}(W_0), d_j = \lambda_{\min}(W_2), j = 1, 2, 3$, and

$$\begin{aligned} a_0 &= u_0 \eta_0 (k_1^{-1} + k_3^{-1}), \\ b_0 &= u_0 \eta_0 (\|A\| + \|B\|) k_1^{-1} + 2 \|B\| k_3^{-1} + u_0 (3 + 2 \|B\| h), \\ c_0 &= \eta_0 u_0 (\|A\| \|B\| k_1^{-1} + \|B\|^2 k_3^{-1}), \\ a_1 &= u_0 \eta_0 \|B\| (k_2^{-1} + k_4^{-1}), \\ b_1 &= u_0 \|B\| (\eta_0 (\|A\| + \|B\|) k_2^{-1} + 2 \|B\| \eta_0 k_4^{-1} + 2), \\ c_1 &= u_0 \eta_0 \|B\| (\|A\| \|B\| k_2^{-1} + \|B\|^2 k_4^{-1}), \\ a_2 &= u_0 (k_1 + k_2 \|B\| h), \\ b_2 &= u_0 (k_1 + k_2 \|B\| h) (\|A\| + \|B\|), \\ c_2 &= u_0 (k_1 + k_2 \|B\| h) \|A\| \|B\|, \\ a_3 &= (1 - \eta_1)^{-1} u_0 (k_3 + k_4 \|B\| h), \\ b_3 &= (1 - \eta_1)^{-1} u_0 2 \|B\| (k_3 + k_4 \|B\| h), \\ c_3 &= (1 - \eta_1)^{-1} \|B\|^2 u_0 (k_3 + k_4 \|B\| h), \end{aligned} \quad (36)$$

the set

$$\mathcal{V} = \left\{ \psi \in \mathcal{C}([-2h, 0], \mathbb{R}^n) : \|\psi\|_{2h} < \sqrt{\frac{\alpha_1}{\alpha_2}} \frac{\delta_2}{2e^{2h(\|A\| + \|B\| + L)}} \right\}, \quad (37)$$

where L is the Lipschitz constant of function $f(u, v)$ on the set

$$\left\{ \psi \in \mathcal{C}([-2h, 0], \mathbb{R}^n) : \|\psi\|_{2h} \leq \frac{\delta_2}{2} \right\} \supset \mathcal{V}$$

and

$$\delta_2 = \frac{2\delta}{2 + h(\|A\| + \|B\| + 2\gamma)},$$

is an estimate of the attraction region for the trivial solution of (21).

Proof. Let the system (3) be exponentially stable. Then, given any $W_j > 0, j = 3, 4, 5$, there exists a unique matrix $U(\cdot)$ satisfying equations (6)-(8), and therefore a unique functional (26) associated to (3). In addition, assume that there exist constants $k_j > 0, j = 1, 2, 3, 4$ such that inequalities (29) hold. Then, from Proposition 2 we have that (23) is exponentially stable and (26) is a Lyapunov-Krasovskii functional for (23). We will show that (26) is a Lyapunov-Krasovskii functional candidate for (32). For the solution $y(t, \varphi)$ of (32), consider $\tilde{y}_t(\varphi) := y(t + \theta, \varphi), \theta \in [-h, 0], t \geq 2h$, and $y_t(\varphi) := y(t + \theta, \varphi), \theta \in [-4h, 0], t \geq 2h$.

The derivative of (26) along solutions of (32), for $t \geq 2h$, is given by:

$$\begin{aligned} \frac{dv(\tilde{y}_t(\varphi))}{dt} &= -w(y_t(\varphi)) + 2[Bz(t, \varphi) + f(y(t, \varphi), y(t-h, \varphi) + z(t, \varphi))]^T \\ &\times \left[U(0)y(t, \varphi) - \int_{-h}^0 U(-h-\theta)By(t+\theta, \varphi)d\theta \right]. \end{aligned} \quad (38)$$

Let

$$\mathcal{U} = \left\{ \varphi \in \mathcal{C}([-4h, 0], \mathbb{R}^n) : \|\varphi\|_{4h} < \frac{\delta_2}{2} \text{ and } v(\tilde{\varphi}) < \alpha_1 \frac{\delta_2^2}{4} \right\},$$

where $\tilde{\varphi}(\theta) = \varphi(\theta)$, $\theta \in [-h, 0]$. For a segment of trajectory $y_t(\varphi) \in \mathcal{U}$, we have $\|y(t+\theta, \varphi)\| < \frac{\delta_2}{2}$, $\theta \in [-4h, 0]$, $t \geq 2h$.

Then $\|(y(t+\theta, \varphi), y(t+\theta-h(t+\theta), \varphi))\| < \delta_2$, $\theta \in [-2h, -h]$, which implies that

$$\|f(y(t+\theta, \varphi), y(t+\theta-h(t+\theta), \varphi))\| \leq \gamma \|(y(t+\theta, \varphi), y(t+\theta-h(t+\theta), \varphi))\|.$$

Therefore

$$\|z(t, \varphi)\| \leq (\|A\| + \gamma) \int_{-h(t)}^{-h} \|y(t+\theta, \varphi)\| d\theta + (\|B\| + \gamma) \int_{-h(t)}^{-h} \|y(t+\theta-h(t+\theta), \varphi)\| d\theta. \quad (39)$$

From (39) we have

$$\|(y(t, \varphi), y(t-h, \varphi) + z(t, \varphi))\| < \frac{\delta_2}{2} (2 + h(\|A\| + \|B\| + 2\gamma)) = \delta,$$

which implies that

$$\|f(y(t, \varphi), y(t-h, \varphi) + z(t, \varphi))\| \leq \gamma \|(y(t, \varphi), y(t-h, \varphi) + z(t, \varphi))\|.$$

Hence

$$\begin{aligned} \|Bz(t, \varphi) + f(y(t, \varphi), y(t-h, \varphi) + z(t, \varphi))\| &\leq (\|B\| + \gamma) (\|A\| + \gamma) \int_{-h(t)}^{-h} \|y(t+\theta, \varphi)\| d\theta \\ &+ \gamma (\|y(t, \varphi)\| + \|y(t-h, \varphi)\|) + (\|B\| + \gamma)^2 \int_{-h(t)}^{-h} \|y(t+\theta-h(t+\theta), \varphi)\| d\theta. \end{aligned}$$

Taking into account this inequality and (16) in (38), we obtain

$$\begin{aligned} \frac{d}{dt}v(\tilde{y}_t(\varphi)) &\leq -w(y_t(\varphi)) + 2u_0 \left[\|y(t, \varphi)\| + \|B\| \int_{-h}^0 \|y(t+\theta, \varphi)\| d\theta \right] \\ &\times \left[\gamma (\|y(t, \varphi)\| + \|y(t-h, \varphi)\|) + (\|B\| + \gamma) (\|A\| + \gamma) \int_{-h(t)}^{-h} \|y(t+\theta, \varphi)\| d\theta \right. \\ &\quad \left. + (\|B\| + \gamma)^2 \int_{-h(t)}^{-h} \|y(t+\theta-h(t+\theta), \varphi)\| d\theta \right]. \end{aligned}$$

The following inequalities hold:

$$\begin{aligned}
& 2u_0\gamma (\|y(t, \varphi)\| + \|y(t-h, \varphi)\|) \left(\|y(t, \varphi)\| + \|B\| \int_{-h}^0 \|y(t+\theta, \varphi)\| d\theta \right) \\
\leq & u_0\gamma \left[(3+2\|B\|h) \|y(t, \varphi)\|^2 + \|y(t-h, \varphi)\|^2 + 2\|B\| \int_{-h}^0 \|y(t+\theta, \varphi)\|^2 d\theta \right], \\
& 2u_0 (\|B\| + \gamma) (\|A\| + \gamma) \left(\int_{-h(t)}^{-h} \|y(t+\theta, \varphi)\| d\theta \right) \left(\|y(t)\| + \|B\| \int_{-h}^0 \|y(t+\theta, \varphi)\| d\theta \right) \\
\leq & u_0 (\|B\| + \gamma) (\|A\| + \gamma) \left[k_2^{-1} \|B\| \eta_0 \int_{-h}^0 \|y(t+\theta, \varphi)\|^2 d\theta \right. \\
& \left. + k_1^{-1} \eta_0 \|y(t, \varphi)\|^2 + (k_1 + k_2 \|B\| h) \int_{-h(t)}^{-h} \|y(t+\theta, \varphi)\|^2 d\theta \right], \\
& 2u_0 (\|B\| + \gamma)^2 \left(\|y(t, \varphi)\| + \|B\| \int_{-h}^0 \|y(t+\theta, \varphi)\| d\theta \right) \int_{-h(t)}^{-h} \|y(t+\theta-h(t+\theta), \varphi)\| d\theta \\
\leq & u_0 (\|B\| + \gamma)^2 \left[k_3^{-1} \eta_0 \|y(t, \varphi)\|^2 + k_4^{-1} \|B\| \eta_0 \int_{-h}^0 \|y(t+\theta, \varphi)\|^2 d\theta \right. \\
& \left. (k_3 + k_4 \|B\| h) \int_{-h(t)}^{-h} \|y(t+\theta-h(t+\theta), \varphi)\| d\theta \right],
\end{aligned}$$

where $k_j, j = 1, 2, 3, 4$ are any free positive constants. Considering inequalities (30) and (31), and after some simple but tedious calculations, we arrive at the following upper bound for the derivative:

$$\begin{aligned}
\frac{dv(\tilde{y}_t(\varphi))}{dt} \leq & -(\lambda_{\min}(W_1) - b\gamma) \|y(t-h, \varphi)\|^2 - [d_0 - (a_0\gamma^2 + b_0\gamma + c_0)] \|y(t, \varphi)\|^2 \\
& - [d_1 - (a_1\gamma^2 + b_1\gamma + c_1)] \int_{-h}^0 \|y(t+\theta, \varphi)\|^2 d\theta \\
& - [d_2 - (a_2\gamma^2 + b_2\gamma + c_2)] \int_{-2h}^{-h} \|y(t+\xi, \varphi)\| d\xi \\
& - [d_3 - (a_3\gamma^2 + b_3\gamma + c_3)] \int_{-4h}^{-2h} \|y(t+\xi, \varphi)\| d\xi, \tag{40}
\end{aligned}$$

where $b = u_0(1 + \|B\|h)$, $a_j, b_j, c_j, j = 0, 1, 2, 3$ are defined by (36), and $d_0 = \lambda_{\min}(W_0)$, $d_j = \lambda_{\min}(W_2), j = 1, 2, 3$.

Hence, if there exists $\gamma > 0$ satisfying $b\gamma < \lambda_{\min}(W_1)$ and the following system of inequalities:

$$a_j\gamma^2 + b_j\gamma + c_j < d_j, \quad j = 0, 1, 2, 3, \tag{41}$$

then the derivative of $v(\tilde{y}_t(\varphi))$ will be negative definite. From the assumption that there exist $k_j > 0, j = 1, 2, 3, 4$ such that inequalities (29) hold, it follows that $d_j > c_j, j = 1, 2, 3$ for the same set of constants $k_j, j = 1, 2, 3, 4$, which implies that the corresponding inequality in (41) is satisfied for any $\gamma \in (0, \gamma_{rj})$, where γ_{rj} is the corresponding positive real root of the equation (35).

Thus, by choosing $\gamma > 0$ such that inequality (34) holds we obtain that $\frac{d}{dt}v(\tilde{y}_t(\varphi)) < 0$, for trajectories of (32) inside the set \mathcal{U} , implying the asymptotic stability of the trivial solution of (32).

Following a similar argument to the one proposed for the proof of theorem 1, we can arrive to prove that the set \mathcal{U} is a positively invariant set with respect to (32) which implies that the set \mathcal{U} is an estimate of the attraction region for the trivial solution of (32).

Let L be the Lipschitz constant of the function $f(\cdot, \cdot)$ on the set

$$\left\{ \psi \in \mathcal{C}([-2h, 0], \mathbb{R}^n) : \|\psi\|_{2h} \leq \frac{\delta_2}{2} \right\} \supset \mathcal{V}.$$

For any initial function $\psi \in \mathcal{V}$, the solution $y(t, \psi)$ of (21) satisfies the following inequality (Halalay 1966):

$$\|y(t, \psi)\| \leq e^{(\|A\| + \|B\| + L)t} \|\psi\|_{2h}, \quad t \geq 0.$$

It follows that

$$\|y_{2h}(\psi)\|_{2h} \leq e^{2h(\|A\| + \|B\| + L)} \|\psi\|_{2h}. \quad (42)$$

Then, the function $\varphi \in \mathcal{C}([-4h, 0], \mathbb{R}^n)$ defined by (33) satisfies

$$\|\varphi\|_{4h} < \sqrt{\frac{\alpha_1}{\alpha_2}} \frac{\delta_2}{2}. \quad (43)$$

Now consider $\tilde{\varphi}(\theta) := \varphi(\theta) = y(2h + \theta, \psi)$, $\theta \in [-h, 0]$. From (43) and the upper bound of $v(\cdot)$, we have

$$v(\tilde{\varphi}) \leq \alpha_2 \|\tilde{\varphi}\|_h^2 < \alpha_1 \frac{\delta_2^2}{4}. \quad (44)$$

From the inequalities (43) and (44) it follows that for any initial function $\psi \in \mathcal{V}$, the function $\varphi \in \mathcal{C}([-4h, 0], \mathbb{R}^n)$ defined by (33) belongs to the set \mathcal{U} , and therefore $y(t, \psi) \rightarrow 0$ when $t \rightarrow \infty$. Hence, the set \mathcal{V} is an estimate of the attraction region for the trivial solution of (21). \square

The estimate of the attraction region (37) depends on the Lipschitz constant of the nonlinear function $f(\cdot, \cdot)$, which result in a more conservative estimate of the attraction region. This is a consequence of using the model transformation to derive such estimate.

Corollary 2 *Let system (3) be exponentially stable and assume that for any given matrices $W_j > 0$, $j = 0, 1, 2$, there exist constants $k_j > 0$, $j = 1, 2, 3, 4$ such that inequalities (29) hold. For any solution $y(t, \psi)$ of (21) such that $\psi \in \mathcal{V}$, the following exponential estimate holds:*

$$\|y(t, \psi)\| \leq \mu \|\psi\|_{2h} e^{-\frac{\beta}{2\kappa} t}, \quad t \geq 0, \quad (45)$$

where $\beta = \min_{j=0,1} \{d_j - (a_j \gamma^2 + b_j \gamma + c_j)\}$ and

$$\mu = \sqrt{\frac{\alpha_2}{\alpha_1}} e^{2h(\|A\| + \|B\| + L + \frac{\beta}{2\kappa})}.$$

Proof. For any initial function $\psi \in \mathcal{V}$, we have that the function $\varphi \in \mathcal{C}([-4h, 0], \mathbb{R}^n)$ defined by (33) belongs to the set \mathcal{U} , and therefore $y_t(\psi) := y(t + \theta, \psi)$, $\theta \in [-h, 0]$, $t \geq 2h$, satisfies that $y_t(\psi) \in \mathcal{U}$, $t \geq 2h$. From inequality (40) it follows that

$$\frac{d}{dt} v(y_t(\psi)) \leq -\beta \left(\|y(t, \psi)\|^2 + \int_{-h}^0 \|y(t + \theta, \psi)\|^2 d\theta \right), \quad t \geq 2h,$$

where

$$\beta = \min_{j=0,1} \{d_j - (a_j \gamma^2 + b_j \gamma + c_j)\}.$$

From (28) we have

$$v(y_t(\psi)) \leq \kappa \left(\|y(t, \psi)\|^2 + \int_{-h}^0 \|y(t + \theta, \psi)\|^2 d\theta \right).$$

It follows that

$$\frac{d}{dt} v(y_t(\psi)) \leq -\frac{\beta}{\kappa} v(y_t(\psi)), \quad t \geq 2h.$$

Integrating this inequality from $2h$ to t we get

$$\ln v(y_t(\psi)) - \ln v(y_{2h}(\psi)) \leq -\frac{\beta}{\kappa} t, \quad t \geq 2h,$$

from which follows

$$\alpha_1 \|y(t, \psi)\|^2 \leq v(y_t(\psi)) \leq v(y_{2h}(\psi)) e^{-\frac{\beta}{\kappa} t} \leq \alpha_2 \|y_{2h}(\psi)\|_h^2 e^{-\frac{\beta}{\kappa} t}, \quad t \geq 2h,$$

and

$$\|y(t, \psi)\| \leq \sqrt{\frac{\alpha_2}{\alpha_1}} \|y_{2h}(\psi)\|_h e^{-\frac{\beta}{2\kappa}t}, \quad t \geq 2h.$$

From (42) we have

$$\|y_{2h}(\psi)\|_h \leq e^{2h(\|A\|+\|B\|+L)} \|\psi\|_{2h}.$$

It follows that

$$\|y(t, \psi)\| \leq \sqrt{\frac{\alpha_2}{\alpha_1}} e^{2h(\|A\|+\|B\|+L)} \|\psi\|_{2h} e^{-\frac{\beta}{2\kappa}t}, \quad t \geq 2h.$$

Since for $t \in [0, 2h]$

$$\|y(t, \psi)\| \leq e^{2h(\|A\|+\|B\|+L)} \|\psi\|_{2h}$$

and $\sqrt{\frac{\alpha_2}{\alpha_1}} \geq 1$, the following inequality holds:

$$\|y(t, \psi)\| \leq \mu \|\psi\|_{2h} e^{-\frac{\beta}{2\kappa}t}, \quad t \geq 0,$$

where

$$\mu = \sqrt{\frac{\alpha_2}{\alpha_1}} e^{2h(\|A\|+\|B\|+L+\frac{\beta}{2\kappa})}.$$

□

5 Examples

In this section, we illustrate our results by some examples from the literature. More precisely, we consider: the stability analysis of a delayed logistic equation (Gopalsamy 1992), the stabilization of nonlinear oscillations by delayed feedback (Niculescu 2001), the stability analysis of Kelly's nonlinear (fluid) first-order delay model for describing congestion phenomena in communication networks (Kelly 2000), and the stability analysis of a nonlinear second-order delay model encountered in some hereditary phenomena in physics (self-excited oscillations in a vacuum tube and theory of stabilization of ships) (Kolmanovskii & Myshkis 1999).

5.1 Delayed Logistic Equation

Let us consider the following delayed logistic equation:

$$\dot{x}(t) = x(t) [r - mx(t) - nx^2(t-h)]. \quad (46)$$

Equation (46) is a generalization of the so-called delayed logistic equation encountered in several problems in biological systems (population dynamics, single species growth model, etc.) see (Gopalsamy 1992). Equation (46) has a unique positive equilibrium point defined by

$$x^* = \frac{-m + \sqrt{m^2 + 4rn}}{2n}.$$

Let $y(t) = x(t) - x^*$, then

$$\dot{y}(t) = ay(t) + by(t-h) + f(y(t), y(t-h)), \quad (47)$$

where $a = -mx^*$, $b = -2n(x^*)^2$, and

$$f(u, v) = -mu^2 - 2nx^*uv - nx^*v^2 - nuv^2.$$

We have

$$|f(u, v)| \leq mu^2 + nx^*(u^2 + v^2) + nx^*v^2 + (n/2)(u^2 + v^2)^{\frac{3}{2}}.$$

Let $\varsigma = \max\{m, nx^*\}$, then

$$|f(u, v)| \leq 2\varsigma(u^2 + v^2) + (n/2)(u^2 + v^2)^{\frac{3}{2}}.$$

Thus, for any given $\gamma > 0$ there exists $\delta(\gamma) > 0$, as the unique real positive solution of the equation $2\zeta\delta + \frac{n}{2}\delta^2 = \gamma$, such that $|f(u, v)| < \gamma \|(u, v)\|$ if $\|(u, v)\| < \delta$. The first approximation

$$\dot{x}(t) = ax(t) + bx(t-h) \quad (48)$$

is exponentially stable for $b < a$ if and only if $h \in [0, h_0)$, where

$$h_0 = \frac{\arccos\left(\frac{-a}{b}\right)}{\sqrt{b^2 - a^2}}.$$

This fact is used in (Gopalsamy 1992) to conclude, by means of a direct application of the stability theorem on the first approximation, that the trivial solution of (47) is asymptotically stable for $h \in [0, h_0)$ in a *sufficiently small neighborhood* of the origin, and then to investigate the existence of periodic solutions of (47) for h close to h_0 . Here, we show how the application of our results can provide an estimate of such neighborhood (the attraction region).

In this case equations (6)-(8) look as

$$\ddot{U}(\tau) = (a^2 - b^2)U(\tau), \quad (49)$$

$$\dot{U}(0) = aU(0) + bU(h), \quad (50)$$

$$-W = 2\dot{U}(0). \quad (51)$$

If equation (48) is exponentially stable, the unique solution of (49), satisfying (50) and (51), is $U(\tau) = Wu(\tau)$, where

$$u(\tau) = \left(\frac{-\lambda + b \sin(\lambda h)}{2\lambda(a + b \cos(\lambda h))} \right) \cos(\lambda\tau) - \frac{1}{2\lambda} \sin(\lambda\tau), \tau \in [0, h]$$

with $\lambda = \sqrt{b^2 - a^2}$, see (Melchor-Aguilar 2004). As an example, consider in (46) that $r = 1$, $m = 1$, and $n = 2$, then the unique positive equilibrium point of (46) is $x^* = 0.5$. We get $a = -0.5$, $b = -1$, and $h_0 = 2.4184$. So, taking $h = 1$ we have that (48) is exponentially stable.

Let $W_0 = 1.5$, $W_1 = 1$ and $W_2 = 0.5$, then $u_0 = 2.4562$. Direct calculations derived from (13) show that

$$0 < \gamma < 0.2036.$$

Selecting $\gamma = 0.2035$, we obtain $\delta = 0.0970$. From (10) and (11) we get $\alpha_1 < 0.75$ and $\alpha_2 \geq 12.8249$. Selecting $\alpha_1 = 0.74$ and $\alpha_2 = 12.8249$, the estimates of the attraction region for the trivial solution of (47) defined by (14) and (20) are given by

$$\mathcal{U} = \{\psi \in \mathcal{C}([-1, 0], \mathbb{R}) : v(\psi) < 0.0017 \text{ and } |\psi|_1 < 0.0485\}$$

and

$$\mathcal{V} = \{\psi \in \mathcal{C}([-1, 0], \mathbb{R}) : |\psi|_1 < 0.0117\}.$$

From (19) we obtain the following exponential bound for any solution starting in the set \mathcal{U} :

$$|y(t, \psi)| \leq 4.1630 |\psi|_1 e^{-1.225 \times 10^{-5}t}, t \geq 0.$$

Now we illustrate the dependence of the attraction region on the delay value. Let us select $h = 0.5$, then (48) is exponentially stable. Let now $W_0 = 1.5$, $W_1 = 0.9$, and $W_2 = 0.6$, then $u_0 = 1.4237$. From (13) we obtain $\gamma < 0.4214$. Choosing $\gamma = 0.4213$, we get $\delta = 0.1922$. From (10) and (11) we obtain $\alpha_1 < 0.75$ and $\alpha_2 \geq 5.0032$. Selecting $\alpha_1 = 0.74$ and $\alpha_2 = 5.0032$, the estimates of the attraction region (14) and (20) are given by

$$\mathcal{U} = \{\psi \in \mathcal{C}([-0.5, 0], \mathbb{R}) : v(\psi) < 0.0068 \text{ and } |\psi|_{0.5} < 0.0961\}$$

and

$$\mathcal{V} = \{\psi \in \mathcal{C}([-0.5, 0], \mathbb{R}) : |\psi|_{0.5} < 0.0370\}.$$

The corresponding exponential bound (19) is given by

$$|y(t, \psi)| \leq 2.6 |\psi|_{0.5} e^{-3.1391 \times 10^{-5}t}, t \geq 0.$$

5.2 Stabilizing nonlinear oscillations by delayed feedback

Let us consider the following nonlinear equation:

$$\ddot{y}(t) = -w_0^2 \sin y(t). \quad (52)$$

Equation (52) is the well-known pendulum equation without friction. This equation could also describe the elevation motion of certain types of helicopter models, see (Roesch *et al.* 2005). Equation (52) have two equilibrium points $(0, 0)$ and $(0, \pi)$. The equilibrium $(0, 0)$ is stable but not asymptotically stable, and it is not possible to stabilize it by a static feedback of the form $u(t) = ky(t)$. Recently, in (Abdallah *et al.* 1993) (see also (Niculescu 2001)), it has been shown that the linear oscillator

$$\ddot{y}(t) = -w_0^2 y(t)$$

can be stabilized by a static delayed feedback of the form

$$u(t) = ky(t - \tau), \quad (53)$$

for all the pairs (k, τ) :

$$0 < k < w_0^2 \text{ and } \underline{\tau}_i(k) < \tau < \bar{\tau}_i(k), \quad (54)$$

where

$$\underline{\tau}_i(k) = \frac{2i\pi}{\sqrt{w_0^2 - k}} \text{ and } \bar{\tau}_i(k) = \frac{(2i+1)\pi}{\sqrt{w_0^2 + k}}, i = 0, 1, \dots$$

Let us rewrite (52) as

$$\ddot{y}(t) = -w_0^2 y(t) + w_0^2 [y(t) - \sin y(t)].$$

Then, the closed-loop (52)-(53) is

$$\ddot{y}(t) = -w_0^2 y(t) + ky(t - \tau) + w_0^2 (y(t) - \sin y(t)). \quad (55)$$

Let $x_1(t) = y(t)$ and $x_2(t) = \dot{y}(t)$, then

$$\dot{x}(t) = Ax(t) + Bx(t - \tau) + f(x(t), x(t - \tau)),$$

where $x(t) = \begin{pmatrix} x_1(t) & x_2(t) \end{pmatrix}^T$, $A = \begin{pmatrix} 0 & 1 \\ -w_0^2 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ k & 0 \end{pmatrix}$, and $f(x(t), x(t - \tau)) = \begin{pmatrix} 0 & w_0^2 [x_1(t) - \sin x_1(t)] \end{pmatrix}^T$

For $0 < u < \frac{\pi}{2}$ or $-\frac{\pi}{2} < u < 0$, the following inequalities hold:

$$1 - \frac{u^2}{2} < \frac{\sin u}{u} < 1.$$

It follows that for any given $0 < \gamma \leq \frac{w_0^2 \pi^2}{8}$ there exists $\delta = \frac{\sqrt{2\gamma}}{w_0}$ such that

$$|w_0^2 [x_1(t) - \sin x_1(t)]| < \gamma |x_1(t)| \text{ if } |x_1(t)| < \delta.$$

Hence, a direct application of the stability theorem on the first approximation leads to conclude that if the pair (k, τ) satisfies (54), then the trivial solution of (55) is asymptotically stable for *sufficiently small initial conditions*.

Considering $w_0^2 = 1$ and $k = 0.5$, the first delay interval guaranteeing closed-loop stability is given by $0 < \tau < 2.5651$. Let $\tau = 1$, $W_0 = 1.5I$, $W_1 = 0.9I$ and $W_2 = 0.3I$. A piece-wise linear approximation of the unique matrix function $U(\cdot)$ solution of (6), satisfying (7) and (8), is plotted in Fig. 1. From Fig. 1 we have $u_0 = 7.1517$. Direct calculations derived from (13) show that

$$0 < \gamma < 0.0839.$$

Selecting $\gamma = 0.0838$, we get $\delta = 0.4094$. From (10) and (11) we obtain $\alpha_1 < 0.6$ and $\alpha_2 \geq 21.4551$. Selecting $\alpha_1 = 0.59$ and $\alpha_2 = 21.4551$, the estimates of the attraction region (14) and (20) are given by

$$\mathcal{U} = \{ \psi \in \mathcal{C}([-1, 0], \mathbb{R}^2) : v(\psi) < 0.0247 \text{ and } \|\psi\|_1 < 0.2047 \}$$

and

$$\mathcal{V} = \{ \psi \in \mathcal{C}([-1, 0], \mathbb{R}^2) : \|\psi\|_1 < 0.0339 \}.$$

From (19) we obtain the following exponential bound for any solution of (55) starting in the set \mathcal{U} :

$$\|y(t, \psi)\| \leq 6.0303 \|\psi\|_1 e^{-1.6030 \times 10^{-5} t}, t \geq 0.$$

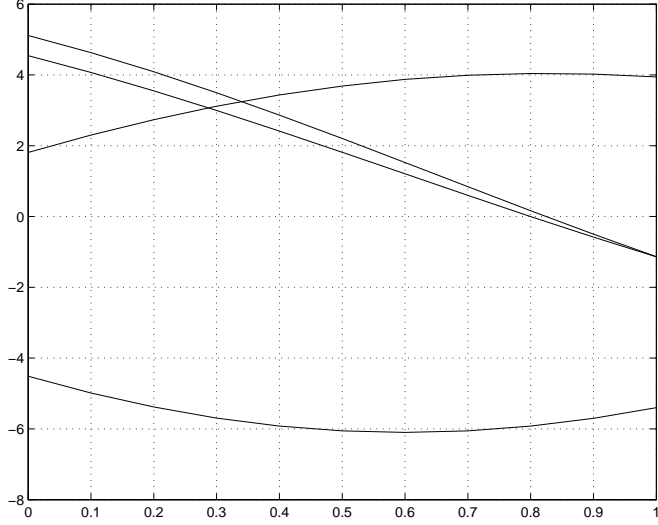


Figure 1: Components of matrix $U(\tau)$

5.3 Congestion control problem in networks

Let us consider the following nonlinear equation:

$$\dot{x}(t) = k [w - x(t - h(t))p(x(t - h(t)))], \quad (56)$$

where k and w are positive reals, $p(\cdot)$ is a continuous and differentiable nondecreasing function, and $h(t)$ is a continuous positive function. Equation (56) describes the dynamics of a collection of flows, all using a single resource and sharing the same gain parameter k , see (Kelly 2000).

The delay function $h(t)$ represents the round-trip time. In the context of congestion control networks, the delay function is usually modeled as $h(t) = h + \eta(t)$, where h is the propagation delay (assumed constant) and $\eta(t)$ represents the queueing delay in the congested router. Here, we assume that $\eta(t)$ is a continuous function satisfying (22). The function $p(\cdot)$ can be interpreted as the fraction of packets indicating (potential) congestion (presence), see (Kelly 2000). Considering that the function $p(x) = k_p x$ with $k_p > 0$, (see (Hollot *et al.* 2002) for a discussion on the benefits to use a proportional controller), equation (56) can be written as

$$\dot{x}(t) = k (w - k_p x^2(t - h(t))). \quad (57)$$

The unique positive equilibrium point of (57) is

$$x^* = \sqrt{\frac{w}{k_p}}.$$

Let $y(t) = x(t) - x^*$, then

$$\dot{y}(t) = by(t - h(t)) + f(y(t - h(t))), \quad (58)$$

where $b = -2k k_p x^*$ and $f(y(t - h(t))) = -k k_p y^2(t - h(t))$. For any given $\gamma > 0$ there exists $\delta = \frac{\gamma}{k k_p}$ such that if $|y(t - h(t))| < \delta$ then

$$|f(y(t - h(t)))| < \gamma |y(t - h(t))|.$$

The first approximation of (58) is

$$\dot{y}(t) = by(t - h(t)). \quad (59)$$

From Proposition 2 we have that the system (59) is exponentially stable if

$$\dot{y}(t) = by(t - h) \quad (60)$$

is exponentially stable and for any given $W_j > 0, j = 0, 1, 2$, there exist $k_j > 0, j = 1, 2, 3, 4$ such that inequalities (29) hold. In this case, inequalities (29) rewrite as:

$$\begin{cases} W_0 > \eta_0 b^2 u_0 k_3^{-1} \\ W_2 > |b|^3 u_0 \eta_0 k_4^{-1} \\ W_2 > \frac{b^2}{1-\eta_1} u_0 (k_3 + k_4 |b| h) \end{cases} \quad (61)$$

The system (60) is exponentially stable if and only if

$$2bh > -\pi \text{ or } k_p < \frac{\pi^2}{16w(hk)^2}.$$

In this case, equations (6)-(8) become:

$$\ddot{U}(\tau) = -b^2 U(\tau), \quad (62)$$

$$\dot{U}(0) = bU(h), \quad (63)$$

$$2\dot{U}(0) = -W. \quad (64)$$

If the system (60) is exponentially stable, then the unique solution of (62) satisfying (63) and (64) is $U(\tau) = Wu(\tau)$, see (Melchor-Aguilar 2004), where

$$u(\tau) = \left(\frac{-1 + \sin(bh)}{2b \cos(bh)} \right) \cos(b\tau) - \frac{1}{2b} \sin(b\tau), \tau \in [0, h].$$

As an example, consider the network parameters from (Kunniyur & Srikant 2003): $k = 0.01, w = 1$, and $h = 0.1$. Taking $k_p = 0.01$, equation (60) is exponentially stable and $x^* = 10$.

Let $W_0 = 0.4, W_1 = 6$, and $W_2 = 0.001$, then $u_0 = 1600$. Let $k_j = 0.0013, j = 1, 2, 3, 4$. From the third inequality in (61) we obtain $\eta_1 < 0.9917$. From the first and second inequalities in (61) we obtain $\eta_0 < 0.0812$.

Thus, a direct application of the stability theorem on the first approximation leads to conclude the local asymptotic stability of the trivial solution of (58) for a continuous time-varying function $\eta(t)$ satisfying

$$0 \leq \eta(t) < 0.0812 \text{ and } 0 \leq \dot{\eta}(t) < 0.9917.$$

Now we show how the application of our results allow us to compute an estimate of the attraction region.

First we observe that for any $\psi_1, \psi_2 \in \{\psi \in \mathcal{C}([-2h, 0], \mathbb{R}) : |\psi|_{2h} \leq \frac{\delta_2}{2}\}$ it holds that

$$|f(\psi_1(\theta)) - f(\psi_2(\theta))| \leq L |\psi_1(\theta) - \psi_2(\theta)|, \theta \in [-2h, 0],$$

where $L = kk_p \delta_2$. Let us select $\eta_0 = 0.08$ and $\eta_1 = 0.9$. Simple calculations derived from (34) show that

$$0 < \gamma < 1.1220 \times 10^{-6}.$$

Choosing $\gamma = 1.12 \times 10^{-6}$, we get $\delta = 0.0112$. From (27) and (28) we obtain $\alpha_1 < 200$ and $\alpha_2 \geq 1760.8$. Selecting $\alpha_1 = 199$ and $\alpha_2 = 1760.8$, the estimate (37) of the attraction region is given by

$$\mathcal{V} = \{\psi \in \mathcal{C}([-0.2, 0], \mathbb{R}) : |\psi|_{0.2} < 0.0019\}.$$

From (45) we obtain the following exponential bound for any solution of (58) starting in the set \mathcal{V} :

$$|y(t, \psi)| \leq 2.9758 e^{-3.3099 \times 10^{-9} t} |\psi|_{0.2}, t \geq 0.$$

Now consider $\eta_0 = 0.01$ and $\eta_1 = 0.9$. Simple calculations derived from (34) show that

$$0 < \gamma < 7.1917 \times 10^{-5}.$$

Selecting $\gamma = 7.19 \times 10^{-5}$, we obtain $\delta = 0.7190$. Thus, the estimate (37) of the attraction region is given by

$$\mathcal{V} = \{\psi \in \mathcal{C}([-0.2, 0], \mathbb{R}) : |\psi|_{0.2} < 0.1208\}$$

and the corresponding exponential estimate (45), for any solution starting in the set \mathcal{V} , is given by

$$|y(t, \psi)| \leq 2.9758 e^{-2.6384 \times 10^{-8} t} |\psi|_{0.2}, t \geq 0.$$

5.4 Hereditary phenomena in physics

Let us consider the following equation:

$$\ddot{x}(t) + 2r\dot{x}(t) + p^2x(t) + 2q\dot{x}(t-1) = \varepsilon\dot{x}^3(t-1). \quad (65)$$

Equation (65) arises in modelling the dynamics of oscillations in a vacuum tube and in the theory of self-excited oscillations, see (Kolmanovskii & Myshkis 1999) and the references therein. This equation is also encountered in the theory of stabilization of ships, see (Kolmanovskii & Myshkis 1999).

Let $y_1(t) = x(t)$ and $y_2(t) = \dot{x}(t)$, then (65) can be written as

$$\dot{y}(t) = Ay(t) + By(t-1) + f(y(t), y(t-1)), \quad (66)$$

where $y(t) = (y_1(t) \ y_2(t))^T$, $A = \begin{pmatrix} 0 & 1 \\ -p^2 & -2r \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & -2q \end{pmatrix}$, and $f(y(t), y(t-1)) = (0 \ \varepsilon y_2^3(t-1))^T$.

The unique equilibrium point of (66) is the origin. For any given $\gamma > 0$ there exists $\delta = \sqrt{\frac{\gamma}{\varepsilon}}$ such that

$$|\varepsilon y_2^3(t-1)| < \gamma |y_2(t-1)| \text{ if } |y_2(t-1)| < \delta.$$

The first approximation of (66) is

$$\dot{y}(t) = Ay(t) + By(t-1). \quad (67)$$

Applying a frequency sweeping test, it is easy to show that the system $\dot{x}(t) = Ax(t) + Bx(t-\tau)$ with A, B given above is *delay-independent exponentially stable* if $2(r^2 + q^2) > p^2$, see (Niculescu 2001). Considering $r = p = q = 1$, we have the exponential stability of the system (67).

Let $W_0 = I$, $W_1 = 0.75I$, and $W_2 = 0.5I$. A piece-wise linear approximation of the unique matrix $U(\cdot)$ solution of (6) satisfying (7) and (8) is plotted in Fig. 2. From Fig. 2 we have $u_0 = 5.0801$. Direct calculations derived from (13) show that

$$0 < \gamma < 0.0492.$$

Selecting $\gamma = 0.049$ and considering $\varepsilon = 0.5$, we obtain $\delta = 0.3130$. From (10) and (11) we get $\alpha_1 < 0.1464$ and $\alpha_2 \geq 63.4618$. Taking $\alpha_1 = 0.146$ and $\alpha_2 = 63.4618$, the estimates of the attraction region defined by (14) and (20) are given by

$$\mathcal{U} = \{\psi \in \mathcal{C}([-1, 0], \mathbb{R}^2) : v(\psi) < 0.0036 \text{ and } \|\psi\|_1 < 0.1565\},$$

and

$$\mathcal{V} = \{\psi \in \mathcal{C}([-1, 0], \mathbb{R}^2) : \|\psi\|_1 < 0.0075\}.$$

The corresponding exponential upper bound (19) for any solution of (66) starting in the set \mathcal{U} is

$$\|y(t, \psi)\| \leq 20.8487 \|\psi\|_1 e^{-3.3808 \times 10^{-5}t}, \quad t \geq 0.$$

6 Conclusion

In this paper, a constructive procedure to compute estimates of the attraction region for some classes of nonlinear time-delay systems having a linear part in their description is proposed. The method developed is along the lines of the finite dimensional case for constructing estimates of the attraction region whenever the linear part of the system is exponentially stable. The approach is constructive and makes use of a Lyapunov-Krasovskii functional *associated* to the exponentially stable linear system. Some examples illustrate the results.

References

[Abdallah *et al.* 1993] ABDALLAH, C.T., DORATO, P., BENITEZ-READ, J. & BYRNE, R. (1993). Delayed positive feedback can stabilize oscillatory systems. *Proc. American Contr. Conf.*, San Francisco, CA, 3106-3107.

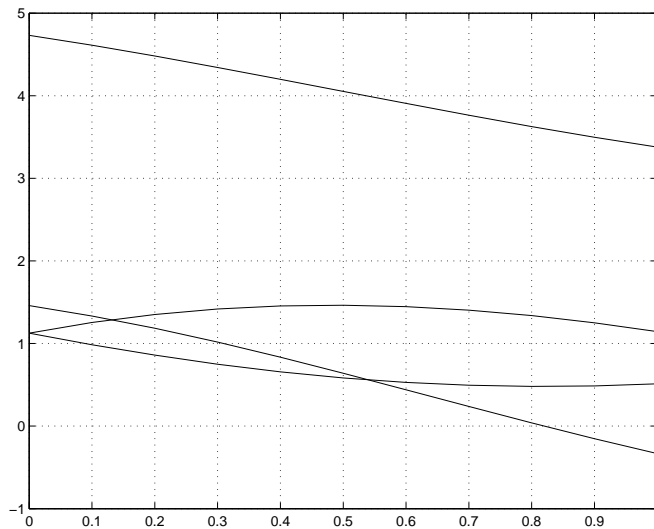


Figure 2: Components of matrix $U(\tau)$

- [Bellman & Cooke 1963] BELLMAN, R. & COOKE, K.L. (1963) *Differential and Difference Equations*. Academic Press.
- [Burton 2006] BURTON, T.A. (2006) *Stability by fixed point theory for functional differential equations*, Dover: Mineola, New York.
- [Gopalsamy 1992] GOPALSAMY, K. (1992) *Stability and Oscillations in Delay Differential Equations of Population Dynamics*. Math. Its Appl. Series, **74**, Kluwer Academic Publishers.
- [Gu & Niculescu 2000] GU, K. & NICULESCU, S.-I. (2000) Additional dynamics in transformed time-delay systems, *IEEE Trans Automat Contr*, **45** (3), 572-575.
- [Gu & Niculescu 2001] GU, K. & NICULESCU S.-I. (2001) Further remarks on additional dynamics in various model transformations of linear delay systems, *IEEE Trans Automat Contr*, **46** (3), 497-500.
- [Halanay 1966] HALANAY, A. (1966) *Differential Equations Stability Oscillations and time lags*. Academic Press.
- [Hale & Verduyn-Lunel 1993] HALE, J.K. & VERDUYN-LUNEL, S.M. (1993) *Introduction to Functional Differential Equations*. Springer Verlag.
- [Hollot *et al.* 2002] HOLLOT, C.V., MISRA V., TOWSLEY, D., & GONG, W. (2002) Analysis and design of controllers for AQM routers supporting TCP flows, *IEEE Trans Automat Contr*, **47** (6).
- [Huang 1989] HUANG, W.(1989), Generalization of the Lyapunov's theorem in a linear delay system, *J. Math. Anal. Appl.*, **182**, 83-94.
- [Infante & Castelan 1978] INFANTE, E.F. & CASTELAN, W.B. (1978) A Lyapunov functional for a matrix difference differential equation, *J. Differ Equations*, **29**, 439-451.
- [Kelly 2000] KELLY, F.P. (2001) Mathematical modelling of the internet, in *Mathematics unlimited - 2001 and beyond* (Eds. B. Engquist, W. Schmid), Springer-Verlag: 685-702.
- [Khalil 1996] KHALIL, H.K. (1996), *Nonlinear systems*, Prentice Hall.
- [Kharitonov & Garcia-Lozano 2004] KHARITONOV, V.L. & GARCIA-LOZANO, H. (2004) Lyapunov matrices for time-delay systems with commensurate delays, Proceedings of 2nd IFAC Symposium on System Structure and Control, Oaxaca, México.

- [Kharitonov & Melchor-Aguilar 2000] KHARITONOV, V.L. & MELCHOR-AGUILAR, D. (2000) On delay-dependent stability conditions, *Syst Control Lett*, **40**, 71-76.
- [Kharitonov & Melchor-Aguilar 2002] KHARITONOV, V.L. & MELCHOR-AGUILAR, D. (2002) On delay-dependent stability conditions for time-varying systems, *Syst Control Lett*, **46**, 173-180.
- [Kharitonov & Niculescu 2003] KHARITONOV, V.L. & NICULESCU, S.-I. (2003) On the Stability of Linear Systems With Uncertain Delay, *IEEE Trans Automat Contr*, **48** (1).
- [Kharitonov & Plischke 2006] KHARITONOV, V.L. & PLISCHKE, E. (2006) Lyapunov matrices for time-delay systems, *Syst Control Lett*, **55**, 697-706.
- [Kharitonov & Zhabko 2003] KHARITONOV, V.L. & ZHABKO, A.P. (2003) Lyapunov-Krasovskii approach to robust stability of time delay systems, *Automatica*, **39**, 15-20.
- [Kolmanovskii & Myshkis 1999] KOLMANOVSKII, V. & MYSHKIS, A. (1999) *Introduction to the theory and applications of functional differential equations*, Kluwer Academic Publishers.
- [Krasovskii 1963] KRASOVSKII, N.N. (1963), *Stability of Motion*, Stanford University Press.
- [Kunniyur & Srikant 2003] KUNNIYUR, S. & SRIKANT, R. (2003) Stable, Scalable, Fair Congestion Control and AQM Schemes that Achieve High Utilization in the Internet, *IEEE Trans Automat Contr*, **48** (11).
- [Lakshmikantham & Leela 1969] LAKSHMIKANTHAM, V. & LEELA, S. (1969) *Differential and Integral Inequalities: Theory and Applications*, Academic Press.
- [Melchor-Aguilar 2004] MELCHOR-AGUILAR, D. (2004) On the Lyapunov's indirect method for scalar differential-difference equations, IFAC Symposium on System Structure and Control, Oaxaca, México.
- [Niculescu 2001] NICULESCU, S.-I. (2001), *Delay effects on stability: A robust control approach*, Lect. Notes Control Inf. Sci., **269**, Springer-Verlag.
- [Repin 1966] REPIN, I.M. (1966) Quadratic Lyapunov functionals for systems with delay, *J. Appl. Math. Mech.*, **29**, 669-672.
- [Richard *et al.* 1997] RICHARD, J.P., GOUBET-BARTHOLOMEUS, TCHANGANI, P.A. & DAMBRINE, M. (1997) Nonlinear Delay Systems: Tools for a Quantitative Approach to Stabilization, in *Stability and Control of Time-Delay Systems*, Lect. Notes Control Inf. Sci., Springer Verlag.
- [Roesch *et al.* 2005] ROESCH, O., HUBERT, R. & NICULESCU S.-I. (2005) Remote Control of Mechatronic Systems over Communication Networks, *Proceedings of IEEE International Conference on Mechatronic and Automation*, Niagara Falls, Canada.
- [Verriest 2000] VERRIEST, E. (2000) Robust Stability and Stabilization: From Linear to Nonlinear, *Proceedings of 2nd IFAC Workshop on Linear Time Delay Systems*, Ancona, Italy.