Preprint of an article published in Journal of Knot Theory and Its Ramifications, 19 (08): 1051-1074, (2010)
https://doi.org/10.1142/S0218216510008327

Braid solutions to the action of the Gin enzyme

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Abstract

The topological analysis of enzymes, a currently active research topic, allowed the deduction of the action mechanism of several enzymes, modelled as 2-string tangles, by an application of the tangle model of Ernst and Sumners. In this article we analyse knotted and linked products of site-specific recombination mediated by the Gin DNA invertase, an enzyme which involves 3-string tangles, and give two families of solutions to its action in both, the directly and inversely repeated sites cases, whenever the 3-tangles involved are 3-braids. For each case, one of the given solutions had not previously been reported in the related literature. In addition, a detailed pseudo-code algorithm is presented which allows one to compute solutions under the assumption that the product of two or more rounds of recombinations are known.

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Keywords: *Knots, 3-braids, Tangle model, Kauffman Bracket Polynomial, enzyme actions.*

1. *Introduction*

Tangles, as mathematical objects, were introduced by J. Conway in [**4**] and have proved to be useful tools in the study of knots and links, for example in the classification of 2-bridge and Montesinos knots [**1**, **13**], and in applications of knot theory to molecular biology [**6**, **9**]. In turn, the tangle model, introduced by Sumners et al. in [**15**], was applied under reasonable biological assumptions to model the site-specific recombinase Tn3 resolvase, as well as other enzymes such as λ-Int [**6**] and Xer [**16**]. It is worth mentioning that such cases involved actions of enzymes on 2-tangles, a quite favourable situation since a complete classification of rational 2-tangles exists in the literature. Some enzymes, however, such as Gin and Hin integrase recombinases, act on 3-tangles instead of 2-tangles. In [**17**], Vazquez and Sumners gave a solution to the action of the Gin enzyme with inversely and directly repeated sites. Their result is based on the application of the tangle model under the additional assumption that one of the strings involved remains completely fixed by Fis, an accessory protein, and hence that the presence of this string can be neglected, thereby obtaining a 2-tangle.

In DNA site-specific recombination, a recombination enzyme attaches to a pair of DNA specific sites, breaks both, and recombines them to different ends, thereby modifying the original topology of the molecule. Electron micrographs of recombinases bound to DNA show the enzyme as a blob from which two or three DNA loops emerge, depending on the enzyme. In the specific case of the Gin enzyme, three DNA loops stick out of the blob, thus making of the theory of 3-tangles a particularly powerful analysis tool.

Regarding 3-tangles, a classification of a subset of the set of *rational* 3-tangles, by means of the Kauffman bracket polynomial and certain invariant matrices, is described in [**2**, **3**]. What is more, the results in [**2**] provide a complete classification of the set $\mathbb{B}_3$ of rational 3-braids—a special case of rational 3-tangles.

Building on the theory and results developed in [**2**], in this paper we exploit the properties of standard braid diagrams and apply the main ideas of the tangle model in order to analyse knotted products of site-specific recombination mediated by the Gin enzyme. As a result, we obtain two solutions to its action in both the directly and inversely repeated cases, under the assumption that the tangles involved are 3-braids. It is interesting to mention that two of the four given solutions, one for each case, were not previously available in the relevant literature. Moreover, we describe an algorithm to compute solutions when the product of two rounds of recombinations are equal to known 2-bridge knots (also referred to as 4-plats). In more mathematical terms, our algorithm permits the solution of equations involving braids whose closures equal the given 2-bridge knots. It is worth mentioning that in [**7**, **8**], an analysis is made of tangle equations which are the numerators of sums of 2-tangles and whose products are either 4-plats or Montesinos knots.

The organisation of the paper is as follows. In Sections 2 and 3 we recall the definition and basic properties of 3-braids, which are special cases of rational tangles, as well as the bracket polynomial applied to tangles. We also give a characterisation of a matrix M_1 associated to a braid. In Section 4, we study some of the properties of a given 3-tangle invariant F, obtained from the matrix M_1. In Section 5 those results are applied to the analysis of the action of the gin enzyme in both the directly and inversely repeated sites cases. Section 6 contains a pseudo-code listing of our algorithm to solve braid equations using the results in previous sections.

2. *Preliminaries*

2·1. *Tangles*

An *n-tangle* is a pair (B^3, T), where B^3 is the 3-ball and T is a set of n disjoint properly embedded arcs in B^3. An n-tangle (B^3, T) is called *rational* if there is a homeomorphism of pairs from (B^3, T) to $(D, P) \times I$, where D is the unit disk, P is any set of n points in the interior of D and I is the closed unit interval. A *tangle diagram* is the projection of the tangle onto the yz-plane.

In this work we shall only deal with rational 3-tangles, and refer to them simply as *tangles*. Accordingly, we shall write T for a tangle (B^3, T). Two n-tangles T and T' are said to be *equivalent*, denoted $T = T'$, if there is a homeomorphism of pairs $h : T \longrightarrow T'$ such that h restricted to ∂B^3 is the identity function or, equivalently, if one can transform T into T' by repeatedly performing Reidemeister moves in B^3 (c.f. Figure 1), keeping ∂B^3 fixed. Given two tangle diagrams TD_1 and TD_2, their sum $TD_1 + TD_2$ is obtained

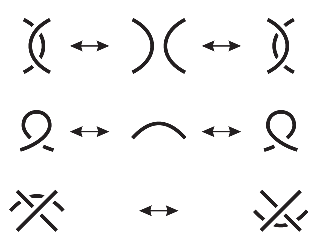


Fig. 1. Reidemeister moves.

by concatenation (or juxtaposition), as shown in Figure 2.

It was shown in [**2**, **3**], by an application of Kauffman's bracket polynomial to tangles, that every tangle diagram TD has five associated Laurent polynomials invariant under

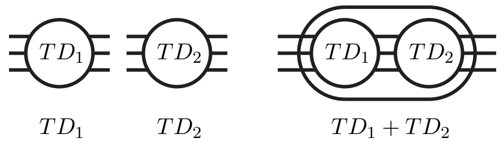


Fig. 2. The diagrams of TD_1, TD_2 and $TD_1 + TD_2$.

regular isotopy. For each tangle diagram TD of a tangle T, these polynomials allow one to define, in turn, two polynomial matrices $M_1(TD)$ and $M_2(TD)$. It turns out that M_2, along with a certain equivalence class $[M_1]$ of M_1, constitute ambient isotopy invariants which provide a classification of a subset of rational 3-tangles, namely the set $\mathbb{B}_3$ of 3-braids. More specifically, given a 3-braid $T \in \mathbb{B}_3$ and its related matrices $M_1(T)$ and $M_2(T)$, one easily computes the standard diagram $AD+kE$ associated with T, consisting of the sum of an alternating diagram AD and k copies of a (nonalternating) 3-braid E. On the other hand, whether such matrix invariants completely classify the set of 3-tangles is still an open question.

2·2. *3-Braids*

An n-*braid* can be defined as a set of n strings attached to vertical bars at their left and right endpoints (see Figure 3(a)), with the property that each string heads rightwards at every point as it is traversed from left to right. The n-braids form a particular class of rational n-tangles (Figure 3(b)). Since in this paper we deal exclusively with 3-braids, we shall drop the prefix '3-' and refer to any of them simply as 'braid.'

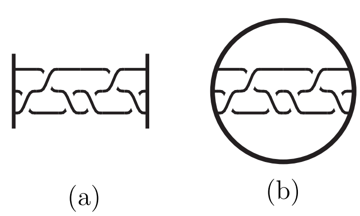
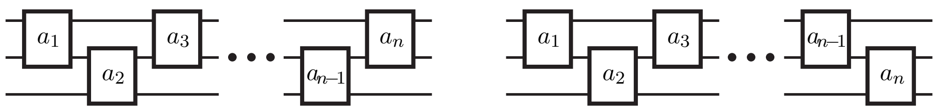


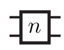
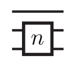
Fig. 3. (a) A 3-braid, and (b) the tangle induced by it.

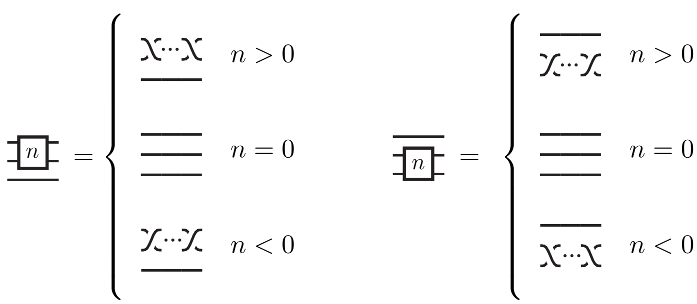
A braid diagram is said to be ***non-alternating*** if any of its strings, as traversed from left to right, exhibits two consecutive overpasses or two consecutive underpasses. A diagram is alternating if it fails to be non-alternating. The braid in Figure 3(a), for instance, is non-alternating. For a detailed introduction to braids see [**12**, **14**].

As a matter of notation, we write $\mathcal{T}(a_1, \ldots, a_n)$, $a_j \in \mathbb{Z}$, to denote either of the following braid diagrams, according to whether n is odd or even:



n odd $\qquad\qquad$ n even

In the above diagrams, each box of the form  or  comprises $|n|$ two-string crossings according to the following conventions:



(2·1)

Note that for any braid T we have an associated diagram $\mathcal{T}(a_1, \ldots, a_n)$ for some $a_j \in \mathbb{Z}$, $j = 1, \ldots, n$. Also note that, with our convention, $\mathcal{T}(a_1, \ldots, a_n)$ is an alternating diagram if, and only if, either $a_i \geq 0$, $i = 1, \ldots, n$ or $a_i \leq 0$, $i = 1, \ldots, n$. An example of an alternating diagram is depicted in Figure 4.

Fig. 4. Example of an alternating diagram: $\mathcal{T}(3, 2, 1, 2, 2)$

An important class of non-alternating braid diagrams is generated by E and $-E$, which are diagrams defined as follows

$$E = \mathcal{T}(1, -1, 1) = \mathcal{T}(0, -1, 1, -1) \quad \text{and} \quad -E = \mathcal{T}(-1, 1, -1) = \mathcal{T}(0, 1, -1, 1).$$

Given $k \in \mathbb{Z}$, we let

$$kE = \begin{cases} E + E + \cdots + E, \ (k \text{ times}), & k > 0; \\ \mathcal{T}(0), & k = 0; \\ -E - E - \cdots - E, \ (|k| \text{ times}), & k < 0. \end{cases}$$

A braid diagram B is said to be ***standard*** if has the form $B = AD + kE$, with AD alternating and $k \in \mathbb{Z}$. It was shown in [**2**] that every braid admits a unique standard diagram:

Lemma 2·1 *For every braid $T \in \mathbb{B}_3$ there exists a unique alternating diagram AD and a unique integer $k \in \mathbb{Z}$ such that a diagram of T equals $AD + kE$.*

The proof is simple and the underlying intuition is exemplified in Figure 5: Starting with a tangle diagram, one "twists" all the strands at places where two consecutive overpasses or underpasses occur, then compensates that twist by another in the opposite direction at the rightmost end of the diagram. Following [**10**], we shall refer to this procedure as a ***flype move***. For 3-tangles, a flype move is related to application of Lagrange's rule (cf. the description of Apply-Lagrange-At in Section 6).

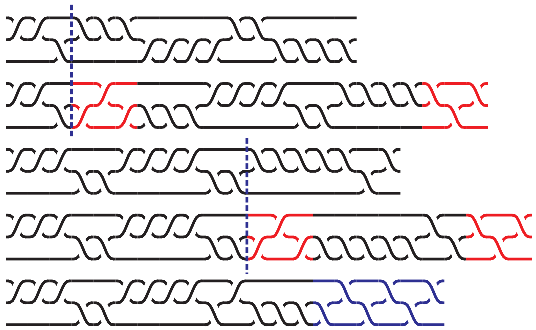
2·3. *Kauffman bracket polynomials and the invariant M_1.*

The Kauffman bracket is a function from unoriented link diagrams to Laurent polynomials with integer coefficients in an indeterminate a. It maps a diagram D to $\langle D \rangle \in \mathbb{Z}[a, a^{-1}]$ and is characterised by the following three conditions:

(K1) $\langle \bigcirc \rangle = 1$

(K2) $\langle TD \sqcup \bigcirc \rangle = -(a^2 + a^{-2}) \langle TD \rangle$

(K3) $\langle \times \rangle = a \langle)(\rangle + a^{-1} \langle \asymp \rangle$



$\mathcal{T}(-2,-1,3,4,2,-4)$

$\mathcal{T}(-2,-1+1,-1,-3+1,-4,-2,4,0,1,-1,1)$

$\mathcal{T}(-3,-2,-4,-2,4,0,1,-1,1)$

$\mathcal{T}(-3,-2,-4,-2+1,-1,-4+1,0,1,-1,1,0,1,-1,1)$

$\mathcal{T}(-3,-2,-4,-1,-1,-3,0,1,-1,1,0,1,-1,1)$

Fig. 5. Deformation of $\mathcal{T}(-2,-1,3,4,2,-4)$ into the standard diagram $\mathcal{T}(-3,-2,-4,-1,-1,-3)+2E$. At each place indicated by a dotted line, a full 3-string twist (show in red) is introduced; in order to preserve the braid, the twist is compensated at the end of the diagram by adding an E summand of the opposite sign (also shown in red). The result of these two *flype moves* appears as a $2E$ summand, shown in blue in the last diagram.

In this definition, $\bigcirc$ is the diagram of the unknot, with no crossings, and $TD \sqcup \bigcirc$ is the diagram consisting of the diagram TD together with an extra closed curve $\bigcirc$ that contains no crossings at all, neither with itself nor with TD. In condition **(K3)**, the formula refers to three link diagrams that are exactly the same except near a point where they differ as indicated. It is well known that the Kauffman bracket satisfies the following relations [**10**]:

(i) $\langle \quad \rangle = -a^3 \langle \quad \rangle$; $\langle \quad \rangle = -a^{-3} \langle \quad \rangle$

(ii) $\langle \quad \rangle = \langle \quad \rangle$

(iii) $\left\langle \quad \right\rangle = \left\langle \quad \right\rangle$

These identities show that the bracket polynomial is an invariant of knots under relations (**ii**) and (**iii**), i.e., it is invariant under regular isotopy.

Given a tangle diagram TD, we define the ***bracket polynomial of*** TD by

$$\langle TD\rangle = \alpha(TD)\left\langle \quad \right\rangle + \beta(TD)\left\langle \quad \right\rangle + \delta(TD)\left\langle \quad \right\rangle + \chi(TD)\left\langle \quad \right\rangle + \psi(TD)\left\langle \quad \right\rangle, \qquad (2\cdot 2)$$

where the coefficients $\alpha(TD)$, $\beta(TD)$, $\delta(TD)$, $\chi(TD)$ and $\psi(TD)$ are polynomials in a and a^{-1} that are obtained by using formulas (K2) and (K3) recursively on TD until one comes up with a diagram containing only combinations of the five tangles that appear in (2·2). Note that these polynomials, any of which may be zero, are also invariant under regular isotopy.

One associates, with a given tangle diagram TD, the following matrix:

$$M(TD)(a) = \begin{pmatrix} \alpha(TD)+\chi(TD) & \beta(TD) & 0 \\ \delta(TD) & \alpha(TD)+\psi(TD) & 0 \\ 0 & 0 & \alpha(TD) \end{pmatrix},$$

where the corresponding polynomial entries are taken from Equation (2·2). Again, $M(TD)(a)$ is an invariant under regular isotopy. We have the following result, proved in [**2**].

Theorem 2·2 [**2**] *Given two tangle diagrams TD_1 and TD_2,*

$$\begin{aligned} M(TD_1+TD_2)(a) &= M(TD_1)(a)M(TD_2)(a) \\ &\quad -(a^2+a^{-2})\begin{pmatrix} \chi_1 & \beta_1 & 0 \\ \delta_1 & \psi_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \chi_2 & \beta_2 & 0 \\ \delta_2 & \psi_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

with $\alpha_i = \alpha(TD_i)$, $\beta_i = \alpha(TD_i)$, $\ldots$ $\psi_i = \psi(TD_i)$, $i = 1, 2$.

With a tangle diagram TD we associate the matrix $M_1(TD) = M(TD)(\sqrt{i})$, which results from evaluating $M(TD)$ at $a = \sqrt{i}$. Since $(\sqrt{i})^2 + (\sqrt{i})^{-2} = 0$, $M_1(TD)$ obviously has the following property:

Lemma 2·3 *For any two tangle diagrams TD_1 and TD_2 one has $M_1(TD_1 + TD_2) = M_1(TD_1)M_1(TD_2)$.*

From Property (i) of the Kauffman bracket it follows that

$$TD_1 = TD_2 \Longrightarrow M(TD_1)(a) = (-a)^{3z} M(TD_2)(a)$$

for some $z \in \mathbb{Z}$ which depends on the number of loops in the diagrams. As a consequence, $M_1(TD)$ is not a tangle invariant. An invariant may be obtained from M_1, however, by considering the following relation on $M_{3\times 3}(\mathbb{C})$:

$$A_1 \sim A_2 \Longleftrightarrow A_1 = (-\sqrt{i})^{3z} A_2 \text{ for some } z \in \mathbb{Z}. \tag{2·3}$$

One easily shows that this is in fact an equivalence relation. Moreover:

Lemma 2·4 *The equivalence class* $[M_1(TD)]$ *is a tangle invariant.*

Clearly, if two tangle diagrams TD_1 and TD_2 are equivalent, then

$$[M_1(TD_1)] = [M_1(TD_2)]$$

and, by Lemma 2·3, we have $[M_1(TD_1 \cdot TD_2)] = [M_1(TD_1)][M_1(TD_2)]$. In the sequel we shall only deal with such equivalence classes, denoting $[M_1(TD)]$ simply by $M_1(TD)$ and enclosing its entries using (square) brackets.

2·4. *Continued Fractions*

Here we briefly recall basic facts on continued fractions, to be used in the remainder of the paper. Given $a_1, \ldots, a_n \in \mathbb{C}$, let

$$\begin{aligned} N[a_1] &= a_1 & D[a_1] &= 1 \\ N[a_1, a_2] &= a_2 N[a_1] + 1 & D[a_1, a_2] &= a_2 D[a_1] \\ N[a_1, a_2, a_3] &= a_3 N[a_1, a_2] + N[a_1] & D[a_1, a_2, a_3] &= a_3 D[a_1, a_2] + D[a_1]. \end{aligned}$$

If we define $N[a_{-1}] = 0, D[a_{-1}] = 1, N[a_0] = 1$, and $D[a_0] = 0$, we have the following recursive formulæ

$$
\begin{aligned}
N[a_1, \ldots, a_n] &= a_n N[a_1, \ldots, a_{n-1}] + N[a_1, \ldots, a_{n-2}], \quad n \geq 1, \\
D[a_1, \ldots, a_n] &= a_n D[a_1, \ldots, a_{n-1}] + D[a_1, \ldots, a_{n-2}], \quad n \geq 1.
\end{aligned}
\tag{2·4}
$$

Note that

$$\frac{N[a_1]}{D[a_1]} = \frac{a_1}{1}, \ \frac{N[a_1, a_2]}{D[a_1, a_2]} = a_1 + \frac{1}{a_2}, \ \frac{N[a_1, \ldots, a_n]}{D[a_1, \ldots, a_n]} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots + \cfrac{1}{a_n}}}.$$

We denote $N[a_1, \ldots, a_n]/D[a_1, \ldots, a_n]$ simply by $[a_1, \ldots, a_n]$, that is,

$$[a_1, \ldots, a_n] = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots + \cfrac{1}{a_n}}},$$

and define $\frac{a}{0} = \infty$, $\infty \cdot a = \infty = \infty + a$, and $\frac{a}{\infty} = 0$, where $a \in \mathbb{C}$ with $a \neq 0$. Using the notation $A_n = [a_1, \ldots, a_n]$, the formulæ in (2·4) take the form

$$NA_n = a_n NA_{n-1} + NA_{n-2}, \quad DA_n = a_n DA_{n-1} + DA_{n-2}, \quad n \geq 1,$$

whence $NA_n/DA_n = A_n$.

An n-tuple $[a_1, \ldots, a_n]$ is said to be a ***continued fraction expansion***; if, moreover, $\mathrm{sign}(a_j) \cdot \mathrm{sign}(a_{j+1}) \geq 0$ for $j = 1, \ldots, n-1$, $[a_1, \ldots, a_n]$ is referred to as a ***strict continued fraction expansion***. As mentioned above, it is easy to see that $\mathcal{T}(a_1, \ldots, a_n)$ is an alternating diagram if, and only if, $[a_1, \ldots, a_n]$ is a strict continued fraction expansion.

2·5. *Computation of M_1 for braids*

Here we develop some results that shall allow us to readily compute the matrix $M_1(T)$ for a braid T.

Lemma 2·5 *For every $n \in \mathbb{Z}$,*

$$M_1(\mathcal{T}(n)) = \begin{bmatrix} 1 & 0 & 0 \\ \frac{n}{i} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \textit{and} \quad M_1(\mathcal{T}(0,n)) = \begin{bmatrix} 1 & \frac{-n}{i} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Proof. One clearly has

$$M_1(\mathcal{T}(0)) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad M_1(\mathcal{T}(1)) = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{i} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad M_1(\mathcal{T}(-1)) = \begin{bmatrix} 1 & 0 & 0 \\ \frac{-1}{i} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus, from Lemma 2·3 we obtain, for $n > 0$,

$$M_1(\mathcal{T}(n)) = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{i} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & 0 & 0 \\ \frac{n}{i} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } M_1(\mathcal{T}(-n)) = \begin{bmatrix} 1 & 0 & 0 \\ \frac{-n}{i} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

Hence

$$M_1(\mathcal{T}(n)) = \begin{bmatrix} 1 & 0 & 0 \\ \frac{n}{i} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In a similar manner, one proves that the given expression for $M_1(\mathcal{T}(0,n))$ is valid as well.

Using the previous notation, in general we have the following Lemma:

Lemma 2·6 *Given $a_1, \ldots, a_n \in \mathbb{C}$, the following statements hold:*

$$\textit{If } n \textit{ is odd, then } \begin{pmatrix} 1 & 0 & 0 \\ a_1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a_2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & 0 & 0 \\ a_n & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} DA_n & DA_{n-1} & 0 \\ NA_n & NA_{n-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

If n is even, then $$\begin{pmatrix} 1 & 0 & 0 \\ a_1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & a_2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\cdots\begin{pmatrix} 1 & a_n & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} DA_{n-1} & DA_n & 0 \\ NA_{n-1} & NA_n & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This lemma may be used to determine $M_1(\mathcal{T}A_n)$, as pointed out by the following result.

Lemma 2·7 *Given $\mathcal{T}(a_1,\ldots,a_n)$ one has:*

$$M_1(\mathcal{T}(a_1,\ldots,a_n)) = \begin{cases} \begin{bmatrix} DA_n^i & DA_{n-1}^i & 0 \\ NA_n^i & NA_{n-1}^i & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} DA_n & \frac{-1}{i}DA_{n-1} & 0 \\ \frac{1}{i}NA_n & NA_{n-1} & 0 \\ 0 & 0 & 1 \end{bmatrix}, & \textit{for odd } n; \\ \begin{bmatrix} DA_{n-1}^i & DA_n^i & 0 \\ NA_{n-1}^i & NA_n^i & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} DA_{n-1} & \frac{-1}{i}DA_n & 0 \\ \frac{1}{i}NA_{n-1} & NA_n & 0 \\ 0 & 0 & 1 \end{bmatrix}, & \textit{for even } n, \end{cases}$$

where $a_i \in \mathbb{Z}$, $A_n = [a_1,\ldots,a_n]$ and $A_n^i = \left[\frac{a_1}{i}, \frac{-a_2}{i}, \ldots, \frac{(-1)^{n+1}a_n}{i}\right]$.

3. *The invariant F*

Recall that, given a tangle T, we have the equivalence class

$$M_1(T) = \begin{bmatrix} \alpha+\chi & \beta & 0 \\ \delta & \alpha+\psi & 0 \\ 0 & 0 & \alpha \end{bmatrix}$$

whereas Lemma 2·7 gives us an easy way to compute the matrix associated to a braid diagram $\mathcal{T}(a_1,\ldots,a_n)$. We shall use these facts in order to obtain another invariant and simplify our tasks. Define a mapping F by

$$F(T) = \left(\frac{M_1(T)_{21}}{M_1(T)_{11}}, \frac{M_1(T)_{22}}{M_1(T)_{12}}\right) = \left(\frac{\delta}{\alpha+\chi}, \frac{\alpha+\psi}{\beta}\right),$$

which is a tangle invariant (in fact, any quotient among linear combinations of the entries of $M_1(T)$ is also an invariant). We have the following Theorem, a direct consequence of Lemma 2·7 and of the properties of continued fractions:

Theorem 3·1 *One has*

$$F(\mathcal{T}(a_1 \ldots, a_n)) = \begin{cases} \left(\frac{1}{i}[a_1, \ldots, a_n], \frac{1}{i}[a_1, \ldots, a_{n-1}]\right), & \textit{for } n \textit{ odd;} \\ \left(\frac{1}{i}[a_1, \ldots, a_{n-1}], \frac{1}{i}[a_1, \ldots, a_n]\right), & \textit{for } n \textit{ even.} \end{cases}$$

Let us remark that, while the continued fractions $[a_1, \ldots, a_n]$, $[a_1, \ldots, a_{n-1}]$ defined by the crossings of a tangle diagram $\mathcal{T}(a_1, \ldots, a_n)$ need not be strict, the invariant F uniquely determines a couple of *strict* continued fractions $[q_1, \ldots, q_{n-1}]$ and $[q_1, \ldots, q_n]$, $q_1, \ldots, q_n \in \mathbb{Z}$, such that if we let $(\frac{1}{i}\frac{a}{b}, \frac{1}{i}\frac{c}{d}) = F(\mathcal{T}(a_1, \ldots, a_n))$ then

$$\left(\frac{1}{i}\frac{a}{b}, \frac{1}{i}\frac{c}{d}\right) = \begin{cases} \left(\frac{1}{i}[q_1, \ldots, q_n], \frac{1}{i}[q_1, \ldots, q_{n-1}]\right), & n \text{ odd;} \\ \left(\frac{1}{i}[q_1, \ldots, q_{n-1}], \frac{1}{i}[q_1, \ldots, q_n]\right), & n \text{ even.} \end{cases}$$

If $\mathcal{T}(a_1, \ldots, a_n)$ is an alternating diagram, then $[a_1, \ldots, a_n]$ and $[a_1, \ldots, a_{n-1}]$ are strict continued fraction expansions. Conversely, a pair of strict continued fraction expansions $[a_1, \ldots, a_n]$ and $[a_1, \ldots, a_{n-1}]$ determine a unique alternating diagram $\mathcal{T}(a_1, \ldots, a_n)$.

Recalling, from Lemma 2·1, that every tangle T may represented in the standard form $T = AD + kE$, the following lemma asserts that $F(T)$ determines the alternating part AD.

Lemma 3·2 *For any two alternating braid diagrams T_1 and T_2, one has $T_1 = T_2$ if, and only if, $F(T_1) = F(T_2)$. In particular $T_1 = T_2$ if, and only if, $M_1(T_1) = M_1(T_2)$.*

Proof. If $T_1 = T_2$, it clearly follows that $F(T_1) = F(T_2)$ and $M_1(T_1) = M_1(T_2)$. To prove the converse implication, from the ordered pair $(\frac{1}{i}\frac{a}{b}, \frac{1}{i}\frac{c}{d})$ we obtain a pair of strict continued fraction expansions $[a_1, \ldots, a_{n-1}]$ and $[a_1, \ldots, a_n]$ which, in turn, uniquely determine an alternating diagram $\mathcal{T}(a_1, \ldots, a_n)$. Since T_1 and T_2 are alternating diagrams, it follows that $T_1 = \mathcal{T}(a_1, \ldots, a_n) = T_2$. On the other hand, $M_1(T_1) = M_1(T_2)$ implies $F(T_1) = F(T_2)$ and hence $T_1 = T_2$.

From the proof of Lemma 3·2 we see that the association of rationals and alternating braid diagrams is reversible in the sense that, given a pair of rationals that are equal to the invariant F of an alternating braid diagram $\mathcal{T}$, the latter may be reconstructed without ambiguity.

We shall now turn to studying the invariant M_1 associated to a non-alternating diagram kE. Using Lemma 2·7 we obtain:

$$M_1(E) = \begin{bmatrix} 0 & \frac{1}{i} & 0 \\ \frac{1}{i} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; M_1(2E) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; M_1(3E) = \begin{bmatrix} 0 & \frac{-1}{i} & 0 \\ \frac{-1}{i} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; M_1(4E) = I_3. \tag{3·1}$$

Since $M_1(4E)$ is the identity matrix, the proof of following Lemma is straightforward:

Lemma 3·3 *For any tangle diagram T,*

$$M_1(T + lE) = M_1(T + mE) \Longleftrightarrow l \equiv m \mod 4.$$

Applying this result to F, and using the expressions for $M_1(E)$ and $M_1(2E)$ one has:

Lemma 3·4 *Let $T \in \mathbb{B}_3$ be such that $F(T) = (\frac{1}{i}\frac{c}{a}, \frac{1}{i}\frac{d}{b})$. Then we have*

$$F(T + nE) = \begin{cases} (\frac{1}{i}\frac{c}{a}, \frac{1}{i}\frac{d}{b}), & n \text{ even;} \\ (\frac{1}{i}\frac{d}{b}, \frac{1}{i}\frac{c}{a}), & n \text{ odd.} \end{cases}$$

Given a braid $B = AD + kE$, Lemma 3·4 and the remarks after Theorem 3·1 imply, in particular, that $F(B)$ determines the alternating part AD as well as whether k is odd or even. Using lemmas 3·2 and 3·4, we get:

Lemma 3·5 *Let be $T_1, T_2 \in \mathbb{B}_3$ then*

(i) *If $F(T_1) = (\frac{1}{i}\frac{c}{a}, \frac{1}{i}\frac{d}{b}) = F(T_2)$, then $T_1 = T_2 + 2kE$ for some $k \in \mathbb{Z}$.*

(ii) *If $F(T_1) = (\frac{1}{i}\frac{c}{a}, \frac{1}{i}\frac{d}{b})$ and $F(T_2) = (\frac{1}{i}\frac{d}{b}, \frac{1}{i}\frac{c}{a})$, then $T_1 = T_2 + (2k+1)E$ for some $k \in \mathbb{Z}$.*

Proof. **(i)** Since $F(T_1) = F(T_2)$, by Lemma 3·4 we have $T_1 = AD + k_1E$ and $T_2 = AD + k_2E$ for some $k_1, k_2 \in \mathbb{Z}$. Since $F(T_1) = F(T_2)$, then k_1 and k_2 are both even or both odd. Therefore

$$T_1 = AD + k_1E = AD + k_2E + (k_1 - k_2)E = T_2 + (k_1 - k_2)E,$$

with $(k_1 - k_2)$ even. The proof of **(ii)** is similar, *mutatis mutandis.*

Remark 3·6

An additional matrix is introduced in [**2**], namely $M_2(TD) = M(TD)(\frac{1+i\sqrt{3}}{2})$. It turns out that M_2 is a tangle invariant with the property that

$$M_2(T + rE) = M_2(T + sE) \iff r = s.$$

Using this fact, one concludes that the matrices $M_1(\cdot)$ and $M_2(\cdot)$ completely classify the braid group $\mathbb{B}_3$, as stated in the following result.

Theorem 3·7 *Given $T_1, T_2 \in \mathbb{B}_3$, one has $T_1 = T_2$ if, and only if, $M_1(T_1) = M_1(T_2)$ and $M_2(T_1) = M_2(T_2)$.*

4. *DNA and 2-bridge knots*

The family of knots and links known as *2-bridge knots* has been largely studied, to the point that it is completely classified. This family is closely related to rational 2-tangles [**1**, **4**, **14**] and, as we will see, to the set of 3-braids as well. A standard diagram of a 2-bridge knot has the form shown in Figure 6, and it can be proved that every 2-bridge knots admits an alternating diagram [**14**]. A 2-bridge knot having a standard regular diagram as in Figure 6, with the exception that signs of crossings follow the opposite convention to that adopted in this paper (cf. 2·1), is said to have ***type*** $b(a, b)$, where $\frac{a}{b} = [b_1, \ldots, b_n]$ is a strict continued fraction expansion and n is odd.

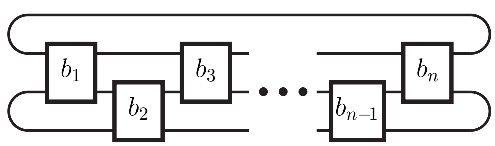


Fig. 6. Standard diagram of a 2-bridge knot.

As mentioned above, 2-bridge knots have been completely classified:

Theorem 4·1 *Suppose that K and K' are 2-bridge knots of types $b(a, b)$ and $b(a', b')$, respectively. Then K and K' are equivalent if, and only if,*

(i) $a = a'$ *and*

(ii) $b \equiv b' \mod a$ *or* $bb' \equiv 1 \mod a$.

Note that, as depicted in Figure 7, the standard diagram of a 2-bridge knot can be regarded as the closure of a braid $\mathcal{T}(a_1, \dots, a_n)$, denoted $A(\mathcal{T}(a_1, \dots, a_n))$, under the proviso that n be odd and $[a_1, \dots, a_n]$ be a strict continued fraction expansion.

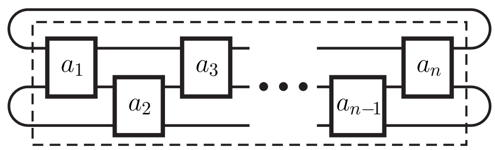


Fig. 7. The closure $A(\mathcal{T}(a_1, \dots, a_n))$ of a tangle diagram $\mathcal{T}(a_1, \dots, a_n)$.

As a consequence of a diagram having non-alternating part equal to $2kE$, we have the following lemma.

Lemma 4·2 *The closure operator A satisfies:*

(i) *For every tangle B and every integer $k \in \mathbb{Z}$, $A(B + 2kE) = A(B)$.*

(ii) *Given $a_2, \dots, a_n \in \mathbb{Z}$, $A(\mathcal{T}(0, a_2, a_3, \dots, a_n)) = A(\mathcal{T}(a_3, \dots, a_n))$.*

Proof. **(i)** follows, by induction on k, from the diagram reduction shown in Figure 8. **(ii)** follows immediately from Figure 7 by letting $a_1 = 0$.

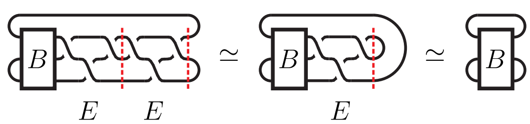


Fig. 8. Illustration of the fact that $A(B + 2E) = A(B)$.

Remark 4·3

According to Lemma 4·2, when taking the A closure of a diagram $\mathcal{T}(a_1, \dots, a_n)$, no generality is lost by assuming, as we shall do in the sequel, that $a_1 \neq 0$ and n is odd. If, moreover, $[a_1, \dots, a_n]$ is a strict continued fraction expansion and $F(\mathcal{T}(a_1, \dots, a_n)) = (\frac{1}{i}\frac{a}{b}, \frac{1}{i}\frac{a'}{b'})$, then $|\frac{a}{b}| \geq 1$.

A relation between 2-bridge knots and 3-braids is described in the following theorem.

Theorem 4·4 *The following hold:*

(i) *Every 2-bridge knot is the A closure of some 3-braid.*

(ii) *The A closure of any 3-braid is a 2-bridge knot, or the unknot, or the 2-component unlink.*

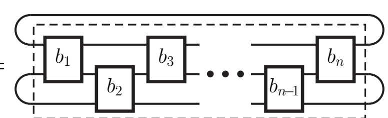
Proof. (i) Assume that a 2-bridge knot K has type (a,b), so that $K = b(a,b)$ with $\gcd(a,b)=1$. Let $[a_1,\dots,a_n]$ (n odd) be a strict continued fraction expansion such that $\frac{a}{-b}=[a_1,\dots,a_n]$ and assume, without loss of generality, that $a_1 \neq 0$ (cf. Remark 4·3). Hence, by definition of type of a 2-bridge knot:

$$b(a,b) = A(\mathcal{T}(a_1,\dots,a_n)).$$

(ii) is obvious.

Lemma 4·5 *Let $B \in \mathbb{B}_3$ and assume that $F(B) = (\frac{1}{i}\frac{a}{b}, \frac{1}{i}\frac{a'}{b'})$ with $1 <|\frac{a}{b}|< \infty$. Then $A(B) = b(a,-b)$.*

Proof. Since a unique strict continued fraction expansion $[b_1,\dots,b_n]$ can be obtained from $F(B)$, from Lemma 3·5 we have $B = \mathcal{T}(b_1,\dots,b_n) + kE$, for some $k \in \mathbb{Z}$. Suppose that n is odd and k even. By Lemma 4·2 we have

$$A(\mathcal{T}(b_1,\dots,b_n)+kE) = A(\mathcal{T}(b_1,\dots,b_n)) =$$ 

Because of the assumption that $|\frac{a}{b}| = |[b_1,b_2,\dots,b_n]| > 1$, one has $b_1 \neq 0$. From the classification of the 2-bridge knots and Lemma 4·2, it follows that

$$A(\mathcal{T}(b_1,\dots,b_n)+kE) = A(\mathcal{T}(b_1,\dots,b_n)) = b(a,-b),$$

where the sign in $-b$ is due to the usual sign convention for 2-bridge knots, [**1**, **14**]. The other cases are analogous.

Using the above lemmas and the classification theorem for 2-bridge knots, we have:

Theorem 4·6 *Let T be a braid such that $F(T) = (\frac{1}{i}\frac{\alpha_1}{\beta_1}, \frac{1}{i}\frac{\alpha_1'}{\beta_1'})$. One has $A(T) = b(\alpha_2, -\beta_2)$ if, and only if,*

(i) $\alpha_1 = \alpha_2$ *and*

(ii) $\beta_1 \equiv \beta_2 \mod \alpha_1$ *or* $\beta_1\beta_2 \equiv 1 \mod \alpha_1$.

Proof. From Lemma 4·5, one gets $A(T) = b(\alpha_1, -\beta_1)$ whereas, by assumption, $A(T) = b(\alpha_1, -\beta_1) = b(\alpha_2, -\beta_2)$. Now, according to the 2-bridge knots classification theorem, the latter equation holds if, and only if, $\alpha_1 = \alpha_2$ and ($\beta_1 \equiv \beta_2 \mod \alpha_1$ or $\beta_1\beta_2 \equiv 1 \mod \alpha_1$).

Note that the previous result does not apply to the unknot; concerning the latter, however, we have:

Theorem 4·7 *Let T be a braid such that $F(T) = (\frac{1}{i}\frac{\alpha_1}{\beta_1}, \frac{1}{i}\frac{\alpha_1'}{\beta_1'})$ with $\mid \frac{\alpha_1}{\beta_1} \mid \geq 1$. Then*

$$A(T) = \textit{Unknot} \iff \left|\frac{\alpha_1}{\beta_1}\right| = 1 \quad \textit{or} \quad \left|\frac{\alpha_1}{\beta_1}\right| = \infty.$$

Proof. Suppose that $A(T)$ is the unknot and that $T = AD + kE$, where $AD = \mathcal{T}(a_1, \ldots, a_n)$ and $[a_1, \ldots, a_n]$ is a strict continued fraction expansion. From Theorem 3·1, Lemma 3·4 and the definition of continued fraction, if $\mathcal{T}(a_1, \ldots, a_n)$ with $n \geq 3$ and $F(\mathcal{T}(a_1, \ldots, a_n)) = (\frac{1}{i}\frac{\alpha_1}{\beta_1}, \frac{1}{i}\frac{\alpha_1'}{\beta_1'})$, then $|\frac{\alpha_1}{\beta_1}| > 1$. By Lemma 4·5, if $1 < |\frac{\alpha_1}{\beta_1}| < \infty$ then $A(B) = b(a_1, -b_1)$ which, by the 2-bridge knots classification theorem, is not the unknot. Whenever $n \leq 2$ we have the following cases:

(i) $F(\mathcal{T}(a_1) + kE) = \big(\infty, \frac{1}{i}[a_1]\big)$ for k odd. Here $|\frac{\alpha_1}{\beta_1}| = \infty$.

(ii) $F(\mathcal{T}(a_1) + kE) = \big(\frac{1}{i}[a_1], \infty\big)$ for k even. Here $|\frac{\alpha_1}{\beta_1}| = 1$ if, and only if, $a_1 = \pm 1$.

(iii) $F(\mathcal{T}(a_1, a_2) + kE) = \big(\frac{1}{i}[a_1, a_2], \frac{1}{i}[a_1]\big)$ for k odd. Here $|\frac{\alpha_1}{\beta_1}| > 1$.

(iv) $F(\mathcal{T}(a_1, a_2) + kE) = \left(\frac{1}{i}[a_1], \frac{1}{i}[a_1, a_2]\right)$ for k even. Here $|\frac{\alpha_1}{\beta_1}| = 1$ if, and only if, $a_1 = \pm 1$.

One readily checks that the A closure of the knots involved in case (i), and in cases (ii) and (iv) when $a_1 = \pm 1$, are equal to the unknot.

5. *Application to the actions of the Gin enzyme*

As mentioned in the introduction, an application of topology to molecular biology is the tangle model for recombination [**9**], where knotted and linked products of site-specific recombination mediated by an enzyme are analysed and, by application of the tangle model, possible solutions to its action are given. Whilst some of the enzymes that have been studied involve rational 2-tangles [**15**, **16**], in this paper we deal with two known actions of the Gin invertase enzyme, corresponding to the cases of *inversely* and *directly repeated sites*, both of which involve 3-tangles. The topological approach to enzymology is the study of enzymes acting on, and thereby modifying the topology of, circular unknotted DNA molecules. Such enzymatic actions are typically produced experimentally by incubating a substrate of circular unknotted DNA and mixing it with a solution of the enzymes under study. A schematic representation of this process is illustrated in Figure 9. For a more detailed and motivated introduction to the topological approach to enzymology and the tangle model, see e.g. [**9**, **15**, **6**].

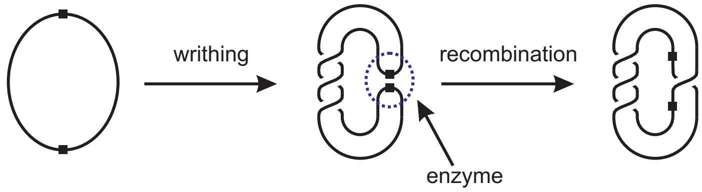


Fig. 9. Illustration of the action of an enzyme acting on a circular, unknotted DNA molecule.

Electron micrographs of the enzyme-DNA complex show the Gin enzyme with three loops of DNA sticking out, which suggests that the classification of rational 3-tangles might be an appropriate tool in order to apply the tangle model. On the other hand, experimental results indicate that under certain biological assumptions, the action of Gin on initially unknotted DNA molecules with inversely repeated sites is a process whose essential topological features are captured by the model depicted in Figure 10, where O, S and T represent 3-tangles. It is reasonable to expect that O, S and T are indeed braids, so we shall assume that it is indeed the case and provide solutions under this assumption.

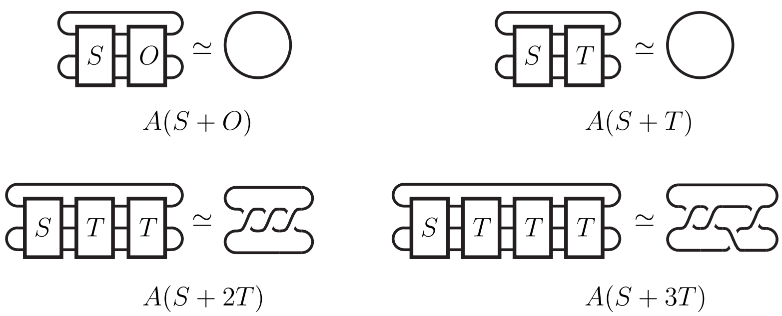


Fig. 10. Repeated action of the Gin enzyme on an initially unknotted DNA molecule.

Using the previous notation, the experimental data in Figure 10 can be translated into the three equations:

$$A(S+T) = \text{Unknot}, \quad A(S+2T) = b(3,1), \quad A(S+3T) = b(5,2). \tag{5·1}$$

In order to solve these equations for S and T, it is convenient to start manipulating the second one. Indeed, by virtue of Theorem 4·6, if X is any braid such that $A(X) = b(3,1)$, its invariant $F(X) = (\frac{1}{i}\frac{a}{b}, \frac{1}{i}\frac{a'}{b'})$ satisfies $a = 3$ and $b \equiv -1 \mod 3$. Now, the condition $|\frac{a}{b}|{>}1$ translates into the constraint $|b| < 3$, which, along with the previous requirements on b, implies that $b = -1$ or $b = 2$. Hence, X should satisfy

$$1)\ F(X) = \left(\frac{1}{i}\frac{3}{-1}, \frac{1}{i}\frac{a'}{b'}\right) \qquad \text{or} \qquad 2)\ F(X) = \left(\frac{1}{i}\frac{3}{2}, \frac{1}{i}\frac{a'}{b'}\right).$$

Since the only strict continued fraction expansions for $\frac{3}{-1}$ are $[-3]$ and $[-2,-1]$ whereas those for $\frac{3}{2}$ are $[1,2]$ and $[1,1,1]$, by Lemma 4·2 we have the following sets (or families) of solutions:

$$X_1 = \mathcal{T}(-3,-a_1) + 2c_1E \qquad X_2 = \mathcal{T}(-2,-1,-a_2) + (2c_2+1)E$$

$$X_3 = \mathcal{T}(1,2,a_3) + (2c_3+1)E \qquad X_4 = \mathcal{T}(1,1,1,a_4) + 2c_4$$

where $a_i, c_j \in \mathbb{Z}$, $a_i \geq 0$. For the third equation we proceed similarly: If a braid Y satisfies $A(Y) = b(5,2)$ and $F(Y) = (\frac{1}{i}\frac{c}{d}, \frac{1}{i}\frac{c'}{d'})$, then it must be that $c = 5$ and ($d \equiv -2 \mod 5$ or $-2d \equiv 1 \mod 5$). Since $|d| < 5$, if $d \equiv -2 \mod 5$, then $d = -2$ or $d = 3$. If, on the other hand, $-2d \equiv 1 \mod 5$, then $d = -3$ or $d = 2$. Hence, in this case our solutions are:

$$Y_1 = \mathcal{T}(-2,-2,-b_1) + (2d_1+1)E \qquad Y_2 = \mathcal{T}(-2,-1,-1,-b_2) + 2d_2E$$

$$Y_3 = \mathcal{T}(1,1,2,b_3) + 2d_3E \qquad Y_4 = \mathcal{T}(1,1,1,1,b_4) + (2d_4+1)E$$

$$Y_5 = \mathcal{T}(2,2,b_5) + (2d_5+1)E \qquad Y_6 = \mathcal{T}(2,1,1,b_6) + 2d_6E$$

$$Y_7 = \mathcal{T}(-1,-1,-2,-b_7) + 2d_7E \qquad Y_8 = \mathcal{T}(-1,-1,-1,-1,-b_8) + (2d_8+1)E,$$

with $b_k, d_l \in \mathbb{Z}$, $b_k \geq 0$.

The previous analysis shows that, if $B_1, B_2 \in \mathbb{B}_3$ satisfy $A(B_1) = b(3,1)$ and $A(B_2) = b(5,2)$, then $B_1 \in X_i$ and $B_2 \in Y_j$ for some $i = 1,2,3,4$ and $j = 1,\ldots,8$. On the other hand, by the tangle model it is assumed that B_1 and B_2 are of the form $S + 2T$ and $S + 3T$, respectively. Therefore, in order to proceed with the solution, for each pair of sets X_i and Y_j we need to find families S_{ij} and T_{ij} such that the following set equalities hold

$$S_{ij} + 2T_{ij} = X_i \quad \text{and} \quad S_{ij} + 3T_{ij} = Y_j.$$

Considering the group structure on $\mathbb{B}_3$, the combination of these two equations yields $X_i + T_{ij} = Y_j$, which in turn implies

$$T_{ij} = -X_i + Y_j \quad \text{and} \quad S_{ij} = X_i + 2(-Y_j + X_i).$$

The latter equations indicate that, for each pair of sets X_i and Y_j, there is a pair of families of solutions S_{ij} and T_{ij}. Performing the required computations, with the aid of the algorithm outlined in Section 6, one comes up with 32 pairs of families S_{ij}, T_{ij} listed in standard form. For each of these pairs, we must check that the first equation in (5·1), i.e., $A(S_{ij} + T_{ij}) = \text{Unknot}$, is actually satisfied. By Theorem 4·7, in order for this equation to hold it is enough to ensure that $|\frac{\alpha_1}{\beta_1}| = 1$ or $|\frac{\alpha_1}{\beta_1}| = \infty$. As applied to the previously obtained solutions, these constraints yield the following families of pairs of solutions

$$\begin{aligned}
&S_1 = \mathcal{T}(0,-2,-1,-a_1) + 2k_1E && T_1 = \mathcal{T}(0,a_1,1,3,a_1) + (2\ell_1+1)E\\
&S_2 = \mathcal{T}(0,-3,-a_2) + (2k_2+1)E && T_2 = \mathcal{T}(0,-a_2,-3,-1,-a_2) + (2\ell_2+1)E\\
&S_3 = \mathcal{T}(-1,-1+a_3) + 2k_3E && T_3 = \mathcal{T}(0,1-a_3,-1-a_3) + (2\ell_3+1)E\\
&S_4 = \mathcal{T}(-1,-2-a_4) + 2k_4E && T_4 = \mathcal{T}(0,2+a_4,a_4) + (2\ell_4+1)E\\
&S_5 = \mathcal{T}(2-a_5) + (2k_5+1)E && T_5 = \mathcal{T}(0,1-a_5,-1-a_5) + (2\ell_5+1)E\\
&S_6 = \mathcal{T}(3+a_6) + (2k_6+1)E && T_6 = \mathcal{T}(0,2+a_6,a_6) + (2\ell_6+1)E,
\end{aligned} \tag{5·2}$$

with $a_i \geq 0$ and $k_i, \ell_i \in \mathbb{Z}$ for $i = 1,\ldots,6$. An interesting fact, as proved in following theorem, is that in addition to solving (5·1), these families solve the equations for *all* rounds of recombinations involved in the tangle model, including the fourth and beyond.

Theorem 5·1 *The pairs of families of solutions* (S_k, T_k) *given in* (5·2) *satisfy, for every* $k \in \{1, \ldots, 6\}$ *and* $n \geq 1$*:*

$$A(S_k + nT_k) = \begin{cases} \text{Unknot}, & n = 1; \\ b(2n-1, 2n-3), & n > 1. \end{cases}$$

Proof. Let $k = 1$. From the definition of $M_1(\cdot)$ we get, by induction on n:

$$M_1(S_1) = \sigma_1 \begin{bmatrix} 3 & -i(3a_1+2) & 0 \\ i & 1+a_1 & 0 \\ 0 & 0 & \sigma_1 \end{bmatrix},$$

$$M_1(nT_1) = \sigma_2 \begin{bmatrix} 2n+1+4a_1n & -ni(1+4a_1(1+a_1)) & 0 \\ -4ni & 1-2n-4na_1 & 0 \\ 0 & 0 & \sigma_2 \end{bmatrix},$$

where σ_1, σ_2 are powers of -1 which depend on the parity of k_1 and ℓ_1, the coefficients of E in the diagrams of S_1 and T_1. As described in Section 3, these matrices allow us to compute $F(S_1 + nT_1)$, the first component of which is given by

$$F(S_1 + nT_1)_1 = \frac{1}{i}\left(\frac{2n-1}{3-2n}\right).$$

According to Theorem 4·6, the A closure of $S_1 + nT_1$ is the 2-bridge knot of type $b(2n-1, 2n-3)$, as stated. The cases corresponding to other choices of k follow by using similar arguments along with Theorem 4·1.

This theorem implies in particular that our solutions satisfy $A(S + 4T) = b(7, 3)$, the (-5)-twist knot. This equation, which is not present in (5·1), is in accordance with the product obtained in the fourth round in an experiment involving the Gin invertase enzyme for a substrate with inversely oriented *gix* sites, [15]. These facts validate the claim that the tangle model based on 3-braids predicts any recombination.

At this point, it is natural to wonder about the smallest number of essentially different families that contain the solutions listed in (5·2). In view of the above theorem, for every $n \in \mathbb{Z}$, the closure $A(S_i + nT_i)$ is independent of the parameters a_i, k_i, ℓ_i, which include the coefficients of E in the standard diagrams of S_i and T_i. One may thus relax the requirement that the a_is be nonnegative or, equivalently, that the diagrams of S_i and T_i be standard, by permitting the a_is to take negative values. As we shall see shortly, this allows one to merge some families, thus reducing the number of different solutions. Another observation is helpful to merge further solutions, to wit, S always appears in (5·2) as the first (leftmost) summand, to which a number of Ts are appended. Therefore, if $S = \mathcal{T}(a_1, a_2, a_3, \dots, a_n) + kE$ "starts" with a zero entry, i.e. $a_1 = 0$, the closure of $S+nT$ is unaffected if one replaces S by $S' = \mathcal{T}(a_3, \dots, a_n)+kE$. These remarks motivate the following definition and lemma.

Definition 5·2 *Let $\sim$ denote the relation defined on pairs of braids by setting $(S,T) \sim (S',T')$ if, and only if, there exist integers $a_0, a_1, \dots, a_n, k_1, k_2$ such that $T = T' + 2k_2E$ and either*

(i) *$S = \mathcal{T}(a_1, \dots, a_n)$ and $S' = \mathcal{T}(0, a_0, a_1, \dots, a_n) + 2k_1E$, or*

(ii) *$S = \mathcal{T}(0, a_0, a_1, \dots, a_n) + 2k_1E$ and $S' = \mathcal{T}(a_1, \dots, a_n)$.*

Lemma 5·3 *The relation $\sim$, as defined above, is an equivalence relation.*

Proof. That $\sim$ is reflexive and symmetric is obvious. To prove transitivity, suppose that $(S,T) \sim (S',T')$ and $(S',T') \sim (S'',T'')$. From the eight possible cases allowed by the definition of $\sim$, we shall only work one out in detail; the rest follow by analogous arguments. Thus, assume that there exist integers $a_0,\ldots,a_n$, $b_0,\ldots,b_m$ k_i,ℓ_i such that $T = T' + 2k_2E$, $T' = T'' + 2\ell_2E$ and both of these conditions hold:

(i) $S = \mathcal{T}(a_1,\ldots,a_n)$ and $S' = \mathcal{T}(0,a_0,a_1,\ldots,a_n) + 2k_1E$,

(ii) $S' = \mathcal{T}(b_1,\ldots,b_m)$ and $S'' = \mathcal{T}(0,b_0,b_1,\ldots,b_m) + 2\ell_1E$.

Clearly, $T = T''+2(\ell_2+k_2)E$. Also, since $S' = \mathcal{T}(0,a_0,a_1,\ldots,a_n)+2k_1E = \mathcal{T}(b_1,\ldots,b_m)$, we conclude that $S'' = \mathcal{T}(0,b_0,0,a_0,a_1,\ldots,a_n)+2(k_1+\ell_1)E = \mathcal{T}(0,b_0+a_0,a_1,\ldots,a_n)+2(k_1+\ell_1)E$. Therefore $(S,T) \sim (S'',T'')$.

An interesting result is that, modulo this equivalence relation, there exist *two* essentially different families of solutions.

Theorem 5·4 *Modulo the relation* $\sim$ *of Definition* 5·2*, there exist two different classes of solutions to the braid equations (*5·1*), namely* (S_α,T_α) *and* (S_β,T_β) *as in Figure* 11*, where* a,b *and* $k_1,\ldots,k_4$ *are integers:*

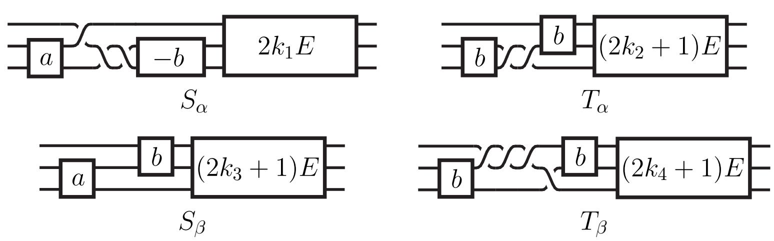


Fig. 11. The two different families of solutions to equations (5·1)

Proof. The proof proceeds in two steps: First we identify families with like classes under $\sim$, denoted with square brackets $[\cdot]$, then prove that the two classes thus obtained are different.

Since $-1 + a_3 = -2 - a_4$ is equivalent to $a_3 = -a_4 - 1$, substituting this value of a_3 in the expressions for S_3 and T_3, we get $[(S_3,T_3)] = [(S_4,T_4)]$. For the third and fifth families, we observe that by taking $\ell_3 = \ell_5$ we get $T_3 = T_5$ and, by application of a flype move:

$$\begin{aligned} S_3 &= \mathcal{T}(-1, -1 + a_3) + 2k_3E \\ &= \mathcal{T}(-1 + 1, -1, 1 - a_3 + 1) + (2k_3 - 1)E \\ &= \mathcal{T}(0, -1, 2 - a_3) + (2k_3 - 1)E. \end{aligned}$$

Setting $a_3 = a_5$ we conclude that $[(S_3, T_3)] = [(S_5, T_5)]$. Similarly, for the fourth and sixth families we take $\ell_4 = \ell_6$ to get $T_4 = T_6$ and then apply a flype move to S_4:

$$\begin{aligned} S_4 &= \mathcal{T}(-1, -2 - a_4) + 2k_4E \\ &= \mathcal{T}(-1 + 1, -1, a_3 + 2 + 1) + (2k_4 - 1)E \\ &= \mathcal{T}(0, -1, a_4 + 3) + (2k_4 - 1)E. \end{aligned}$$

Letting $a_4 = a_6$ we conclude that $[(S_4, T_4)] = [(S_6, T_6)]$. Thus (S_i, T_i) belong to the same class for $i = 3, \ldots, 6$. Note that

$$\begin{aligned} [(S_4, T_4)] &= [\mathcal{T}(-1, -2 - a_4) + 2k_4E, \mathcal{T}(0, 2 + a_4, a_4) + (2\ell_4 + 1)E)] \\ &= [(\mathcal{T}(0, a, -1, -2 - a_4) + 2k_4E, \mathcal{T}(0, 2 + a_4, a_4) + (2\ell_4 + 1)E] \\ &= [(S_\alpha, T_\alpha)], \end{aligned}$$

that is, the class of the first family in Figure 11. Now, applying a flype move to an element of S_2 we obtain

$$
\begin{aligned}
S_2 &= \mathcal{T}(0,-3,-a_2) + (2k_2+1)E \\
&= \mathcal{T}(0,-3+1,-1,a_2+1) + (2k_2+2)E \\
&= \mathcal{T}(0,-2,-1,a_2+1) + 2(k_2+1)E,
\end{aligned}
$$

which is checked to belong to family S_1 by taking $a_1 = -a_2 - 1$ and $k_1 = k_2 + 1$. Substituting the corresponding value $a_2 = -a_1 - 1$ in the expression for T_2, and applying further flype moves we get:

$$
\begin{aligned}
T_2 &= \mathcal{T}(0,a_1+1,-3,-1,a_1+1) + (2\ell_2+1)E \\
&= \mathcal{T}(0,a_1+1-1,1,3-1,1,-a_1-1) + 2\ell_2 E \\
&= \mathcal{T}(0,a_1,1,2,1,-a_1-1) + 2\ell_2 E \\
&= \mathcal{T}(0,a_1,1,2+1,-1,-1+1,a_1+1) + (2\ell_2+1)E \\
&= \mathcal{T}(0,a_1,1,3,-1,0,a_1+1) + (2\ell_2+1)E \\
&= \mathcal{T}(0,a_1,1,3,a_1) + (2\ell_2+1)E,
\end{aligned}
$$

clearly an element of T_1. Thus $[(S_1,T_1)] = [(S_2,T_2)]$. Since the first box in the diagram of S_2 equals 0, we get

$$
\begin{aligned}
[(S_2,T_2)] &= [(\mathcal{T}(-a_2) + (2k_2+1)E, T_2)] \\
&= [(\mathcal{T}(0,a,b) + (2k_2+1)E, \mathcal{T}(0,b,-3,-1,b) + (2\ell_2+1)E)] \\
&= [(S_\beta, T_\beta)],
\end{aligned}
$$

that is, the second family in Figure 11.

To prove that the two families are different, consider $(\mathcal{T}(b)+E, \mathcal{T}(0,b,-3,-1,b)+E)$, a representative of $[(S_\beta, T_\beta)]$. In particular, for $b=0$ we get $(\mathcal{T}(0)+E, \mathcal{T}(0,0,-3,-1,0)+E) = (E, \mathcal{T}(-3,-1)+E) \in [(S_\beta, T_\beta)]$. We shall show that, for every b and k_2 in $\mathbb{Z}$, one has $\mathcal{T}(-3,-1)+E \neq B(b,k_2) := \mathcal{T}(0,b+2,b)+(2k_2+1)E$ so that $[(E,\mathcal{T}(-3,-1)+E)] \neq [(S_\alpha, T_\alpha)]$. To this aim, note that $\mathcal{T}(-3,-1)+E$ is in standard form, and so is $B(b,k_2)$ if $b<-2$ or $b>0$; differing in their standard diagrams, these braids are therefore unequal. Now, if $b=-1$,

$$\begin{aligned} B(-1,k_2) &= \mathcal{T}(0,1,-1)+(2k_2+1)E \\ &= \mathcal{T}(0,1-1,1,1-1)+2k_2E \\ &= \mathcal{T}(1)+2k_2E, \end{aligned}$$

which is in standard form. Thus $\mathcal{T}(-3,-1)+E \neq B(-1,k_2)$. Analogously, if $b=-2$, then $B(-2,k_2) = \mathcal{T}(-2)+(2k_2+1)E$ and hence $\mathcal{T}(-3,-1)+E \neq B(-2,k_2)$. This finishes the proof.

5·1. *Gin acting on substrates with directly repeated sites*

It was shown in [**11**] that, under certain conditions, the Gin enzyme also acts in substrates with directly repeated sites, a case which was analysed in [**17**] under the assumption that the tangles involved were 2-string tangles. In order to apply our algorithm based on 3-braids, and in view of the experimental data, the equations to be considered in this case are

$$A(S+T) = b(3,1), \quad \mathrm{A}(S+2T) = b(7,3) \quad \text{and} \quad A(S+3T) = b(11,9). \tag{5·3}$$

Applying the methodology described above, one finds that the families of solutions are

$$
\begin{array}{ll}
S_1 = \mathcal{T}(0,-2,-1,-a_1) + 2k_1 E & T_1 = \mathcal{T}(0,a_1,1,1,1,2,a_1) + (2\ell_1+1)E \\
S_2 = \mathcal{T}(0,-3,-a_2) + (2k_2+1) E & T_2 = \mathcal{T}(0,-a_2,-2,-1,-1,-1,-a_2) + (2\ell_2+1)E \\
S_3 = \mathcal{T}(-1,-1+a_3) + 2k_3 E & T_3 = \mathcal{T}(0,1-a_3,-1,-1,-a_3) + (2\ell_3+1)E \\
S_4 = \mathcal{T}(-1,-2-a_4) + 2k_4 E & T_4 = \mathcal{T}(0,1+a_4,1,1,a_4) + (2\ell_4+1)E \\
S_5 = \mathcal{T}(2-a_5) + (2k_5+1) E & T_5 = \mathcal{T}(0,1-a_5,-1,-1,-a_5) + (2\ell_5+1)E \\
S_6 = \mathcal{T}(3+a_6) + (2k_6+1) E & T_6 = \mathcal{T}(0,1+a_6,1,1,a_6) + (2\ell_6+1)E \\
S_7 = \mathcal{T}(0,-1,-1,-a_7) + 2k_7 E & T_7 = \mathcal{T}(0,a_7,1,1,1,2,a_7) + (2\ell_7+1)E \\
S_8 = \mathcal{T}(0,-2,-a_8) + (2k_8+1) E & T_8 = \mathcal{T}(0,-a_8,-2,-1,-1,-1,-a_8) + (2\ell_8+1)E,
\end{array}
\tag{5·4}
$$

with $a_i \geq 0$ and $k_i, \ell_i \in \mathbb{Z}$ for $i = 1, \ldots, 8$. An argument analogous to the one in the proof of Theorem 5 shows that, modulo the relation $\sim$ from Definition 5·2, these families fall into the two classes $(S_\alpha, 2T_\alpha)$ and $(S_\beta, 2T_\beta)$ depicted in Figure 12, where S_α, S_β, T_α and T_β correspond to the solutions found in the previous case.

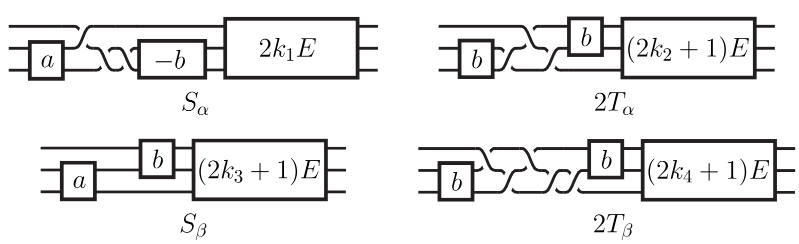


Fig. 12. The two different families of solutions to equations (5·3)

6. *Algorithm*

The following is an algorithm to solve tangle equations using the results described above. Specific implementations (in a variety of computer languages) may easily be obtained from the listed pseudo-code, which adheres to the conventions set forth in [**5**]. As an additional convention, if A is an array, $A[j\,..\,k]$ denotes the (finite) sequence $A[j], A[j+1], \ldots, A[k]$. Due to space limitations, the procedures do not include any data validation or exception handling. Here is a brief description of the procedures involved:

- Append(A, x) Appends the element x at the end of array A.

- CONCATENATE(A, B) Concatenates arrays A and B, in that order.
- GET-CF-EXPANSION$(n, \textit{long-format})$ Gets the continued fraction expansion of a rational n. The argument *long-format* indicates whether the long or short format is required.
- SOLVE-2BK-CONGRUENCES(α, β) Returns an array of couples of integers $[a, b]$ that solve the two-bridge knot congruences: $a = \alpha$ and ($b \equiv \beta \mod \alpha$ or $b\beta \equiv 1 \mod \alpha$).
- GET-SOLUTION-EXPANSION(α, β, X) Returns an array of braids. Each braid B in standard form and satisfies $(F(B))_1 \geq 1$. These braids represent the totality of solutions to equation $A(B) = b(\alpha, -\beta)$.
- APPLY-LAGRANGE-AT$(F, k, \textit{plus-to-minus})$ Applies Lagrange's rule to the continued fraction F, represented as an array of integers, at position $k \in \{1, \dots, \textit{length}[F]\}$. If *plus-to-minus* $=$ TRUE, the rule $a + \frac{1}{-b} = a - 1 + \frac{1}{1 + \frac{1}{b-1}}$ is applied, so that if $F = [f_1, \dots, f_{k-1}, a, -b, f_{k+2}, \dots, f_n]$, it returns $[f_1, \dots, f_{k-1}, a-1, 1, b-1, -f_{k+2}, \dots, -f_n]$. If *plus-to-minus* $=$ FALSE, the rule $-a + \frac{1}{b} = -a + 1 + \frac{1}{-1 + \frac{1}{-b+1}}$ is applied, so that if $F = [f_1, \dots, f_{k-1}, -a, b, f_{k+2}, \dots, f_n]$, it returns $[f_1, \dots, f_{k-1}, -a + 1, -1, -b + 1, -f_{k+2}, \dots, -f_n]$.
- STRIP-ZEROES(F) Returns an array that differs from the array of integers F in that it contains no zero entry.
- REMOVE-ZEROES(F) Takes a continued fraction F, represented as an array, then removes zeros and simplifies according to the rules of continued fractions.
- DETECT-SIGN(P, X) Takes a degree-one multinomial $P = a_0 + a_1 X[1] + \cdots + a_{\textit{length}[X]} X[\textit{length}[X]]$ in the indeterminates $X[i]$ and returns a structure indicating whether P is sign-definite and, in such case, $\operatorname{sign}(P)$. Sign-definiteness is tested under the assumption that the indeterminates take strictly positive values.
- DETECT-SIGN-CHANGE(A, I, X) Takes an array A of multinomials in the indeterminates X and, starting from the Ith position, determines whether there is a sign change. In such case, the position p at which it occurred, and the sign of $A[p]$ are also returned. Sign-detection is based on the rules applied by DETECT-SIGN.
- GET-STANDARD-FORM(B, X) Takes a braid B, given by $\mathcal{T}(\textit{fraction}[B]) + \textit{index}[B]E$ and puts it in the standard form $AD + kE$. *fraction*$[B]$ is an array on multinomials in the indeterminates X.
- CONCATENATE-BRAIDS(A, B, X) Takes braids A and B (cf. GET-STANDARD-FORM (B, X) for the description of braids) and concatenates (i.e., adds) them, expressing the result in standard form.
- INVERT-BRAID(A) Inverts braid A (cf. GET-STANDARD-FORM(B, X) for the description of braids) under the concatenation operation in the braid group.
- GET-P-AND-Q(A, B) Given two 2-bridge knots $b(A[1], A[2])$ and $b(B[1], B[2])$, returns braid families P and Q such that $A(P+Q) = b(A[1], A[2])$ and $A(P+2Q) = b(B[1], B[2])$.

APPEND(A, x)

return $[A[1\,..\,\textit{length}[A]], x]$

CONCATENATE(A, B)

return $[A[1\,..\,\textit{length}[A]], B[1\,..\,\textit{length}[B]]]$

```
GET-CF-EXPANSION(n, long-format)
R ← sign(n)GETCONTINUEDFRACTION(|n|)
if long-format
    then R[length[R]] ← R[length[R]] − sign(n):
         R ← APPEND(R, sign(n))
return R
```

```
SOLVE-2BK-CONGRUENCES(α, β)
T ← {β}
if |α + β| < α
    then T ← T ∪ {α + β}
if | − α + β| < α
    then T ← T ∪ {−α + β}
for ℓ ← −|β| to |β|
      do b ← (α/β)ℓ + 1/β
         if b ∈ ℤ and |b| < |α|
           then T ← T ∪ {b}
create array R of size [1 . . length[T]]
for i ← 1 to length[R]
      do a[R[i]] ← α
         b[R[i]] ← T[i]
return R
```

```
GET-SOLUTION-EXPANSION(α, β, x)
C ← SOLVE-2BK-CONGRUENCES(α, β)
R ← [ ]
for i ← 1 to length[C]
      do
         F ← a[C[i]] / b[C[i]]
         for j ← 1 to 2
              do
                 if j = 1
                    then fraction[S] ← GET-CF-EXPANSION(F, FALSE)
                    else  fraction[S] ← GET-CF-EXPANSION(F, TRUE)
                 fraction[S] ← APPEND(fraction[S], sign(F)x)
                 index[S] = length[fraction[S]]  mod 2
                 R ← APPEND(R, S)
return R
```

```
APPLY-LAGRANGE-AT(F, k, plus-to-minus)
n ← length[F]
if n < 2 or k ∉ {1, . . . , n − 1}
    then return F
if plus-to-minus
    then return [F[1 . . k − 1], F[k] − 1,   1, −F[k + 1] − 1, −F[k + 2 . . n]]
    else  return [F[1 . . k − 1], F[k] + 1, −1, −F[k + 1] + 1, −F[k + 2 . . n]]
```

```
STRIP-ZEROES(F)
R ← [ ]
for i ← 1 to length[F]
     do if F[i] ≠ 0
          then R ← APPEND(R, F[i])
return R
```

```
REMOVE-ZEROES(F)
 1  R ← F
 2  last-zero ← 3
 3  while length[R] > 1 and any entry of R[last-zero . . length[R] − 1] equals 0
 4        do n ← length[R]
 5           for i ← last-zero −1 to n − 1
 6               do if R[i] = 0
 7                     then R ← [R[1 . . i − 2], R[i − 1] + R[i + 1], R[i + 2..n]]
 8                          last-zero ← max{i, 3}
 9                          break     ▷ Breaks the for loop and jumps to 3
10  if length[R] > 1 and R[length[R]] = 0
11     then R ← [R[1.. length[R] − 1]]
12  return R
```

```
DETECT-SIGN(P, X)
    ▷ P is assumed to be given by P = a_0 + a_1 X[1] + ··· + a_{length[X]} X[length[X]]
    ▷ It is assumed that sign(n) = 0 if and only if n = 0
 1  s = [sign(a_0), . . . , sign(a_{length[X]})]
 2  s ← STRIP-ZEROES(s)
 3  if length[s] = 0
 4     then is-definite[R] ← TRUE
 5          sign[R] ← 0
 6     else
 7          m ← min{s[1 . . length[s]]}
 8          M ← max{s[1 . . length[s]]}
 9          if m = M
10             then is-definite[R] ← TRUE
11                  sign[R] ← m
12             else is-definite[R] ← FALSE
13                  sign[R] ← 0
14  return R
```

```
DETECT-SIGN-CHANGE(A, I, X)
 1  sign-changed[R] ← FALSE
 2  position[R] ← 0
 3  plus-to-minus[R] ← FALSE
 4  if length[A] ≤ 1
 5     then return R
 6  create array s of size [1 . . length[A] − I + 1]
 7  for i ← I to length[A]
 8        do s[i − I + 1] ← DETECT-SIGN(A[i], X)
 9  for i ← 2 to length[s]
10        do if is-definite[s[i]]
11              then
12                   ▷ Detect first A[j] to the left of A[i] with definite sign and compare
13                   for j ← i − 1 downto 1
14                       do if is-definite[s[j]] and sign[s[j]] ≠ 0
15                             then
16                                  if sign[s[j]] ≠ sign[s[i]]
17                                    then sign-changed[R] ← TRUE
18                                         position[R] ← i + I − 1
19                                         if sign[s[j]] > sign[s[i]]
20                                            then plus-to-minus[R] ← TRUE
21                                         return R
22  return R
```

GET-STANDARD-FORM(B, X)

```
n ← length[fraction[B]]
F ← REMOVE-ZEROES(fraction[B])
a ← 0
ℓ ← 4
c ← DETECT-SIGN-CHANGE(F, ℓ − 3, X)
while sign-changed[c]
      do F ← REMOVE-ZEROES(APPLY-LAGRANGE-AT(F, position[c] − 1, plus-to-minus[c]))
         if plus-to-minus[c]
           then a ← a + (−1)^{position[c]}
           else a ← a − (−1)^{position[c]}
         ℓ ← max{position[c], 4}
         c ← DETECT-SIGN-CHANGE(F, ℓ − 3, X)
fraction[R] ← F
index[R] ← index[B] + a
return R
```

CONCATENATE-BRAIDS(A, B, X)

```
f_a ← fraction[A];   f_b ← fraction[B];   n_a ← length[f_a];   n_b ← length[f_b]
index[R] ← index[A] + index[B]
if index[A] ∈ 2ℤ
   then if n_a ∈ 2ℤ
           then fraction[R] ← CONCATENATE(f_a, f_b)
           else fraction[R] ← [f_a[1 . . n_a − 1], f_a[n_a] + f_b[1], f_b[2 . . n_b]]
   else if n_a ∈ 2ℤ
           then fraction[R] ← [f_a[1 . . n_a − 1], f_a[n_a] − f_b[1], −f_b[2 . . n_b]]
           else fraction[R] ← CONCATENATE(f_a, −f_b)
return GET-STANDARD-FORM(R)
```

INVERT-BRAID(A)

```
n ← length[fraction[A]]
k ← index[A]
F ← [ ]
if k ∈ 2ℤ
   then if n ∈ 2ℤ
           then F ← [0]
        F ← APPEND(F, [−fraction[A][n . . 1]])
   else if n ∈ 2ℤ + 1
           then F ← [0]
        F ← APPEND(F, [fraction[A][n . . 1]])
fraction[R] ← REMOVE-ZEROES(F)
index[R] ← −k
return R
```

GET-P-AND-Q(A, B)

```
Xs ← GET-SOLUTION-EXPANSION(A[1], A[2], a)
Ys ← GET-SOLUTION-EXPANSION(B[1], B[2], b)
create array R of size [1 . . length[Xs], 1 . . length[Ys]]
for i ← 1 to length[Xs]
      do for j ← 1 to length[Ys]
               do Q ← CONCATENATE-BRAIDS(INVERT-BRAID(Xs[i]), Ys[j], [a, b])
                  P ← CONCATENATE-BRAIDS(Xs[i], INVERT-BRAID(Q), [a, b])
                  X[R[i, j]] ← Xs[i]
                  Y[R[i, j]] ← Ys[j]
                  P[R[i, j]] ← P
                  Q[R[i, j]] ← Q
return R
```

Acknowledgements

This research was partially funded by grants from CONACYT, Fdo. Inst. 66912 S-3124 and 66912 S-3122, as well as CONCYTEG No. 06-02-K117-84.

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