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A generalized design methodology for the output feedback regulation of a special type of systems with bounded inputs

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SUMMARY

In this work, a generalized design scheme for the output feedback regulation of a special type of systems with bounded inputs is proposed. It gives rise to a simple dynamic controller that guarantees the regulation objective avoiding input saturation, for any initial condition within a specific set that may comprehend the whole state space, and that does not require any additional system data (apart from the output variable). Several processes, like double-pipe heat exchangers, bioreactors, and binary distillation columns, are shown to be part of the type of systems that may be regulated through the developed methodology. The efficiency of the proposed scheme is corroborated through experimental and simulation results. Copyright © 2011 John Wiley & Sons, Ltd.

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1. INTRODUCTION

When one is charged to design a control algorithm for an actual application, one is inexorably faced to deal with the inherent complexity and limitations of the plant/process (to be controlled). For instance, if every system internal variable (state) may take any value within a wide range, a linearized model is not necessarily —and will not in general constitute— a suitable approximation. Intrinsic nonlinear phenomena may be stimulated, giving rise to unexpected behaviors —such as sustained oscillatory or vibratory responses, trajectory undesired convergence, or chaotic regimes [1, §1.1]— that may result in unacceptable or even dramatic consequences. Another relevant aspect to take care about is the saturation nonlinearity that usually characterizes the signal transfer from the controller output to the process input. This is a consequence of the natural limitations of real-life actuators. Forcing these to go beyond their natural capabilities, undergoing saturation, may produce undesirable phenomena [2, §5.2], [3, §15.4], [4] that give rise to a deteriorated or even disastrous closed-loop performance, as pointed out for instance in [5] and [6].

Another aspect that is worth considering in the synthesis procedure concerns the process data included in the designed scheme. For instance, the stated control objective may not necessarily involve all the system states but just one or some of them. A controller that achieves the desired goal exclusively through such variables turns out to be convenient. In a natural way, this reduces

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sensor/signal-availability concerns; this is particularly important when the involved variables are the only ones that are measurable. Besides, by avoiding the extraction of unnecessary information from the system, the inherent instrument inaccuracies and measurement noise have a diminished influence on the feedback loop. Less corrupted control signals consequently take place decreasing the related effects and risks. As a matter of fact, the more dependent is the designed scheme on the process data, the more deteriorated closed-loop performance may be expected since it is practically impossible to avoid errors in the model structure, estimated parameters, and/or variable measurements. From this perspective, the lowest possible or null dependence on the exact knowledge of the system parameters turns out to be convenient too.

On the other hand, the control scheme may be designed on the basis of the process intrinsic dynamical properties. A synthesis procedure that exploits the open-loop system inherent analytical features (such as BIBO stability, passivity, or input-to-state stability) generally gives rise to a successful simple algorithm. The antithesis of such a design guideline seems to be established by the so-called *exact linearization via feedback* [7]. Assuming the availability of the exact process information —precise model, unbiased parameters, and accurate state measurements— such a methodology aims at compensating the system dynamics to impose a linear closed loop with *suitable* stability properties. This is done whether or not the open loop already possessed dynamical features that were compatible or appropriate to the control objective. As a result, such a methodology generally gives rise to complex controllers that involve all (or an important amount of) the process data, whose success is highly dependent on the accuracy of the involved system information, and that do not essentially deal with the inherent limitations such as the input saturation phenomenon.

An additional important aspect concerns the region of attraction (of the desired equilibrium) that is achieved in closed loop. In this direction, an algorithm that guarantees the control objective for any initial condition within the system state space, or in a subset of interest (where the closed loop trajectories would evolve globally in time), constitutes the best option.

In this paper, we aim at contributing a generalized control scheme for the output feedback regulation of a special type of bounded-input systems, taking into account the previously mentioned important aspects and guidelines. In particular, we focus on a type of SISO[†] plants for which any constant input value, within its permissible bounded range, ensures the convergence of the system (internal) variables towards a specific point within their domain; examples of such kind of processes comprehend double-pipe heat exchangers [8], bioreactors [9], and binary distillation columns [10], among others. Such an open-loop property is exploited by the proposed scheme, which aims at reproducing a similar closed-loop behavior, inducing the input to converge to the natural value that ensures the desired stabilization, through an auxiliary state-space dynamics whose vector field prevents the control signal to go beyond the corresponding (plant) input natural bounds. This idea was followed in [11] to design an output feedback regulator for double-pipe heat exchangers. In this particular case —which constituted the main motivation to develop the generalized scheme proposed in the present work—, the auxiliary dynamics forces the relocation of the system steady state in such a way that the output variable takes the desired equilibrium value —which is achieved by directly involving the output error (with respect to its desired value) and no additional system data—; simultaneously, equilibrium values are imposed to the auxiliary state at the physical bounds such that the control signal is prevented to go beyond such limits; moreover, the corresponding closed-loop equilibrium points are proved to be unstable in view of which convergence to any of them is avoided; in addition, the auxiliary dynamics include a control gain through which suitable closed-loop properties are ensured. The present work aims at the generalization of this design procedure for its application to other types of systems that keep analogous open-loop characteristics.

A controller similar to that in [11], but that does not involve a control gain (or equivalently, with a fixed unitary control gain), was previously proposed in [9] for the output feedback regulation of bioreactors. A remarkable closed-loop stability proof was developed in this work through Lyapunov

[†]It is considered here that the dynamic model of a SISO system may involve several external variables, but only one of them plays the role of control input —for the stabilization of a particular output variable— while the rest of them remain constant (including those which may be considered perturbation inputs).

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analysis. Later on, the same control technique, this time involving a control gain, was applied for the stabilization of continuous stirred tank reactors in [12]. In this work, the existence of such a stabilizer was proved in a generalized context for a type of SISO systems whose open-loop stability properties (under the consideration of constant input values) were characterized through the existence of a Lyapunov function. The controller auxiliary dynamics obtained through the proof of this result (presented in [12, §4] as Lemma 1), keeps a structure analogous to that in [11], imposing equilibrium values that prevent the auxiliary state to go beyond the input bounds, and involving a specific function of the states that forces the desired stabilization for any (positive) value of the control gain. But if the output error is involved instead to induce the desired stabilization, the control gain is not necessarily free to take any positive value. Indeed, a careful reading of the proof of the main result in [11] shows that with a sufficiently large control gain, the unique equilibrium point in the closed-loop state-space domain becomes unstable, which generates an unexpected asymptotical behavior (such as a sustained oscillation). This particular aspect is reflected in the generalized context considered in this work, which finishes up by restricting the control gain tuning (such a restriction is further relaxed under additional requirements).

Thus, the control scheme proposed in this work is not only a generalization of that developed in [11], but may also be seen as an extension or complement of those presented in [9] and [12]. Firstly, the study focuses on the characterization of the analytical properties of the type of systems that may be regulated through the proposed method. Further, the closed-loop system is proved to keep analytical properties similar to those characterizing the open-loop system, with the unique equilibrium point suitably relocated in terms of the desired output value. Moreover, applicability of the control algorithm is shown on several processes. The proposed scheme proves to achieve the output feedback regulation objective avoiding input saturation, without the need of any additional system data (apart from the output variable), and for any state initial condition within a specific set where the system is known to satisfy the requested characterization (and which could comprehend the whole natural state space of the considered plant).

The paper is organized as follows. Section 2 states the considerations, assumptions, and notations used throughout the paper. The main result is presented in Section 3. Section 4 treats the application of the proposed scheme to double-pipe heat exchangers, bioreactors, and binary distillation columns; results obtained through experimental implementations on a UASB bioreactor and other simulation tests are included. Finally, conclusions are given in Section 5.

2. CONSIDERATIONS, ASSUMPTIONS, AND NOTATION

Let $\mathbb{R}^n_+$ represent the set of vectors in $\mathbb{R}^n$ whose elements are all nonnegative, 0_n denote the origin of $\mathbb{R}^n$, and I_n stand for the $n \times n$ identity matrix. The i^{th} element of a vector $\xi \in \mathbb{R}^n$ is denoted ξ_i. The interior and boundary of a set, say Γ, are respectively denoted $\text{int}(\Gamma)$ and $\partial\Gamma$. Let $\mathcal{A}$ and $\mathcal{E}$ be subsets (with nonempty interior) of some vector spaces $\mathbb{A}$ and $\mathbb{E}$ respectively. The image of $\mathcal{B} \subset \mathcal{A}$ under $\nu : \mathcal{A} \to \mathcal{E}$ is denoted $\nu(\mathcal{B})$. Differentiability of $\nu : \mathcal{A} \to \mathcal{E}$ at any point of the boundary of $\mathcal{A}$ (when included in the set) is meant as the limit from the interior of $\mathcal{A}$. For a continuously differentiable scalar function $\nu : \mathcal{A} \to \mathbb{R}$, where $\mathcal{A} \subset \mathbb{R}$, we denote $\nu' : \mathcal{A} \to \mathbb{R} : \varsigma \mapsto \frac{d\nu}{d\varsigma}$, *i.e* $\nu'(\varsigma) = \frac{d\nu}{d\varsigma}(\varsigma)$. As conventionally, the inverse of an invertible function ν is denoted ν^{-1}. For a symmetric matrix $B \in \mathbb{R}^{n\times n}$, $\lambda_{\text{min}}(B)$ and $\lambda_{\text{max}}(B)$ respectively denote its minimum and maximum eigenvalues. For a matrix $B \in \mathbb{R}^{m\times n}$, $\|B\|$ represents the standard Euclidean induced matrix norm (or 2-norm), *i.e.* $\|B\| = \left[\lambda_{\text{max}}(B^T B)\right]^{1/2}$; with $m = n$, *i.e.* $B \in \mathbb{R}^{n\times n}$, $\det(B)$ denotes the determinant of B, and $B > 0$ expresses that B is positive definite.

Let us consider a dynamical system with state model of the form

$$\dot{x} = f(x) + g(x)u \triangleq F(x, u) \tag{1a}$$

$$y = h(x) \tag{1b}$$

where $x \in \mathbb{R}^n$; $u \in [\underline{u}, \bar{u}] \triangleq \Upsilon$, for some constants $\underline{u} < \bar{u}$; $y \in \mathbb{R}$; $f, g : \mathbb{R}^n \to \mathbb{R}^n$; and $h : \mathbb{R}^n \to \mathbb{R}$. The following assumptions characterize the type of systems considered in this study.

A1. f, g, and h are continuously differentiable.
A2. There exists a continuously differentiable function $\psi : \mathbb{R} \to \mathbb{R}^n$ satisfying: $F(\psi(u), u) = 0_n$, $\forall u \in \Upsilon$.
A3. There is a compact set $\Omega \subset \mathbb{R}^n$ containing the image of Υ under ψ, *i.e.* $\Psi \triangleq \psi(\Upsilon) \subset \Omega$, that is positively invariant with respect to (1a) uniformly in u on Υ.
A4. The Jacobian matrix $\frac{\partial F}{\partial x}(\psi(u), u)$ is Hurwitz for all $u \in \Upsilon$.
A5. With $u = \alpha$, for any constant $\alpha \in \Upsilon$:

(a) invariant sets on $\partial\Omega$, if any, are unattractive;
(b) $\{\psi(\alpha)\}$ is the unique invariant set in $\mathrm{int}(\Omega)$.

A6. The scalar function $\phi(u) \triangleq h(\psi(u))$, is monotonic on Υ.

Remark 1
Let $u = \alpha$ for any constant $\alpha \in \Upsilon$. Under this consideration, let $x(t; x_0)$ represent the solution of system (1a) with initial condition $x(0; x_0) = x_0$.

R1. Assumptions A1 and A3 ensure existence and uniqueness of solutions of system (1a) for any $x_0 \in \Omega$ [1, Theorem 3.3 & Lemma 3.2]. In particular, by the additional consideration of Assumption A5a, we have that, for any $x_0 \in \mathrm{int}(\Omega)$, $x(t; x_0) \in \mathrm{int}(\Omega)$, $\forall t \geq 0$.
R2. Assumptions A2–A5 state that there exists a unique equilibrium point $\psi \in \mathrm{int}(\Omega)$, whose location in $\mathrm{int}(\Omega)$ is determined by the value of $\alpha \in \Upsilon$, that such unique equilibrium in $\mathrm{int}(\Omega)$ is asymptotically stable —actually, exponentially stable [1, Theorem 4.15]— whatever is the value that α takes in Υ, and that it constitutes the unique invariant in $\mathrm{int}(\Omega)$.
R3. From points R1 and R2 above, and the compact character of Ω (which implies boundedness of every trajectory remaining in Ω), we have that, for any $x_0 \in \mathrm{int}(\Omega)$, $x(t; x_0) \to \psi(\alpha)$ as $t \to \infty$ —see for instance [1, Lemma 4.1]—, and consequently: $y(t) \to h(\psi(\alpha))$ as $t \to \infty$.
R4. Observe from Assumption A6 that:

(a) the image of Υ under ϕ, $\mathcal{R} \triangleq \phi(\Upsilon)$, is given by $\mathcal{R} = [\phi(\underline{u}), \phi(\bar{u})]$ if $\phi(u)$ is increasing, or by $\mathcal{R} = [\phi(\bar{u}), \phi(\underline{u})]$ if $\phi(u)$ is decreasing; more generally $\mathcal{R} = \left[\min\{\phi(\underline{u}), \phi(\bar{u})\}, \max\{\phi(\underline{u}), \phi(\bar{u})\}\right]$; this shows that the output cannot be regulated to any real value but only to those in $\mathcal{R}$;
(b) any output steady-state value $\phi \in \mathcal{R}$ is uniquely defined by a specific input fixed value $u \in \Upsilon$; moreover, this in turn implies, since $\phi = h \circ \psi$, that every output steady-state value $\phi \in \mathcal{R}$ is uniquely related to a specific equilibrium vector $\psi \in \Psi$, which is in turn uniquely defined by a specific input fixed value $u \in \Upsilon$;
(c) ϕ, as a function mapping Υ onto $\mathcal{R}$, is invertible; as a function mapping $\mathrm{int}(\Upsilon)$ onto $\mathrm{int}(\mathcal{R})$, it is actually diffeomorphic (the differentiability is a consequence of Assumptions A1 and A4; in particular, ϕ' is obtained in Appendix A).

Remark 2
Under Assumptions A1–A3, observe that Assumption A5b is satisfied if there exists a continuously differentiable scalar function $V(x; \alpha)$ such that $\dot{V}(x; \alpha) = \frac{\partial V}{\partial x} F(x, \alpha) \leq 0$, $\forall (x; \alpha) \in \Omega \times \Upsilon$, with $\dot{V}(\psi(\alpha); \alpha) = 0$, $\forall \alpha \in \Upsilon$, and $\dot{V}(x; \alpha) < 0$, $\forall (x; \alpha) \in \mathrm{int}(\Omega) \times \Upsilon \setminus \{(\psi(\alpha), \alpha)\}$ (according to La Salle's invariance principle [1, Theorem 4.4] and the satisfaction of Assumption A5a). Assumption A5b holds as well under the satisfaction of the conditions stated by Rosenbrock's Theorem [10, Appendix A], which was developed and thoroughly proven in [13] (specifically stated as Theorem 6 in the appendix of this reference). Actually, the systems that satisfy Rosenbrock's Theorem are a special case of those considered in this work. Other ways to prove the satisfaction of Assumption A5b can be developed for systems with simple models, as seen for instance in [8] for a heat exchanger bi-compartmental model, and later on in this work (§4.1 and, more specifically, Appendix C) for a simple model of a bioreactor.

Basically, Assumptions A1–A6 state that the system under consideration is such that (with constant input) every trajectory with initial condition within the interior of a positively invariant compact set, denoted Ω, —which could comprehend the whole system state space— converges

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to a unique equilibrium point whose location is determined by the value of u; moreover, that the structural stability of the system is not lost whatever value take u in Υ. Such a characterization comprehends —but is not restricted to— processes whose dynamics keeps the system trajectories evolving within a bounded set beyond which the system states have no physical meaning; the systems considered later on in Section 4 are actually of this kind. Simultaneously, systems whose states may take any real value are also included; this is the case for instance when the asymptotic stability of the unique open-loop equilibrium point ψ is global. Indeed, from [1, Theorem 4.17] one sees that for such systems, there exists a Lyapunov function $V_\psi(x)$ that proves the asymptotic stability of ψ, defined in the whole region of asymptotic stability R^A_ψ, and such that the compact set $\{V_\psi(x) \leq c\}$ is contained in R^A_ψ for any $c > 0$. Thus, Assumptions A1–A6 are satisfied with $\Omega = \{V_\psi(x) \leq c\}$ for any $c > \underline{c}$, for a suitable value of $\underline{c}$ (such that $\Psi \subset \{V_\psi(x) \leq \underline{c}\}$). This holds whether R^A_ψ is equal to $\mathbb{R}^n$ or not. But the developed characterization is stated in such a way that either a specific Ω with the required properties is known —without any restriction on how such a subset should be found or defined (as long as it possesses the required properties)—, or whose existence is qualitatively known (even without a precise definition), like in the globally asymptotically stable equilibrium case. In this latter case, the proposed scheme achieves the regulation objective for any initial condition on the system internal variables (since c may take any value arbitrarily higher than $\underline{c}$).

3. MAIN RESULT

Observe that with $u = \alpha$ for any constant $\alpha \in \Upsilon$, the considered type of system, with any $x_0 \in \operatorname{int}(\Omega)$, performs a natural stabilization to the unique equilibrium $\psi(\alpha) \in \operatorname{int}(\Omega)$, which corresponds to a natural output regulation to $h(\psi(\alpha)) = \phi(\alpha) \in \mathcal{R}$ (according to point R4b of Remark 1). This gives rise to the idea of designing a dynamic controller through which the closed-loop dynamics keeps the same analytical features in $\Omega \times \Upsilon$, with u forced to evolve within $\operatorname{int}(\Upsilon)$, and forcing the existence of a unique equilibrium point $(x^*_d, u^*_d) = (\psi(u^*_d), u^*_d)$ (and no other invariant set in the interior of $\Omega \times \Upsilon$), strategically located such that the corresponding steady-state output value be equal to the (pre-specified) desired value $y_d \in \operatorname{int}(\mathcal{R})$, *i.e.* such that $h(x^*_d) = h(\psi(u^*_d)) = \phi(u^*_d) = y_d$. This is achieved through the following control scheme.

Proposition 1

Consider the dynamical system in Eqs. (1) under Assumptions A1–A6, with continuity of the m^{th} order partial derivatives of f and g holding up to $m = 2$. Let the input be defined as

$$u = \theta \tag{2}$$

with θ being an auxiliary state whose dynamics is defined as

$$\dot{\theta} = sk\eta(\theta)(y_d - y) \tag{3}$$

for any (constant) $y_d \in \operatorname{int}(\mathcal{R})$, where $s = \operatorname{sign}\big(\phi(\bar{u}) - \phi(\underline{u})\big)$, k is a positive constant, and η is a continuously differentiable scalar function satisfying $\eta(\underline{u}) = \eta(\bar{u}) = 0$, $\eta'(\underline{u}) > 0 < \eta'(\bar{u})$, and $\eta(\theta) > 0$, $\forall \theta \in (\underline{u}, \bar{u})$. Then, provided that k is sufficiently small, for any initial condition $(x, \theta)(0) = (x_0, \theta_0) \in \operatorname{int}(\Omega) \times \operatorname{int}(\Upsilon)$: $(x, \theta)(t) \to \varphi(y_d) \triangleq (\psi(\phi^{-1}(y_d)), \phi^{-1}(y_d))$ as $t \to \infty$, and consequently $y(t) \to y_d$ as $t \to \infty$, with $u(t) \in \operatorname{int}(\Upsilon)$, $\forall t \geq 0$, and $x(t) \in \operatorname{int}(\Omega)$, $\forall t \geq 0$.

Proof

Let $z = (z^T_{(n)}, z_{n+1})^T \in \mathbb{R}^{n+1}$, with $z_{(n)} = (z_1, \ldots, z_n) \in \mathbb{R}^n$, denote the extended state vector, *i.e.* $z = (z^T_{(n)}, z_{n+1})^T \triangleq (x^T, \theta)^T$. The closed-loop state model adopts the form

$$\dot{z} = \bar{f}(z) + \bar{g}(z)y_d \triangleq \bar{F}(z; y_d) \tag{4a}$$

$$y = \bar{h}(z) \tag{4b}$$

with

$$\bar{f}(z) = \begin{pmatrix} f(z_{(n)}) + g(z_{(n)})z_{n+1} \\ -sk\eta(z_{n+1})h(z_{(n)}) \end{pmatrix} \tag{4c}$$

$$\bar{g}(z) = \begin{pmatrix} 0_n \\ sk\eta(z_{n+1}) \end{pmatrix} \tag{4d}$$

and

$$\bar{h}(z) = h(z_{(n)}) \tag{4e}$$

In analogy to Assmptions A1–A6, the following points of the proof are enumerated.

P1. From Assumption A1 and the auxiliary subsystem dynamics in Eq. (3), $\bar{f}$, $\bar{g}$, and $\bar{h}$ happen to be continuously differentiable.

P2. From the auxiliary subsystem dynamics in Eq. (3) and point R4 of Remark 1, one sees that $\varphi(y_d) \triangleq \left(\psi^T(\phi^{-1}(y_d)), \phi^{-1}(y_d)\right)^T$ satisfies $\bar{F}(\varphi(y_d); y_d) = 0_{n+1}$, $\forall y_d \in \mathcal{R}$. Observe that, from the continuous differentiability of ψ and point R4c of Remark 1, φ turns out to be continuously differentiable on $\mathrm{int}(\mathcal{R})$.

P3. Let us begin by noting, from Assumption A3 and point R4 of Remark 1, that $\varphi(\mathcal{R}) \subset \Omega \times \Upsilon$. Now, since $\eta(\underline{u}) = \eta(\bar{u}) = 0$, $\theta = \underline{u}$ and $\theta = \bar{u}$ happen to be equilibrium values of the auxiliary subsystem (3). From this and the positively invariant character that Ω keeps with respect to (1a) uniformly in u on Υ (according to Assumption A3), one sees that, for any $(x_0, \theta_0) \in \Omega \times \Upsilon$, the closed-loop system solutions cannot leave $\Omega \times \Upsilon$. Hence, $\Omega \times \Upsilon$ is a compact set, containing the image of $\mathcal{R}$ under φ, that is positively invariant with respect to (4a).

P4. From Eqs. (1)–(4), we have that

$$\frac{\partial \bar{F}}{\partial z}(z; y_d) = \begin{pmatrix} \frac{\partial F}{\partial x}(z_{(n)}, z_{n+1}) & g(z_{(n)}) \\ -sk\eta(z_{n+1})\frac{\partial h}{\partial x}(z_{(n)}) & sk\eta'(z_{n+1})\big(y_d - h(z_{(n)})\big) \end{pmatrix} \tag{5}$$

The rest of this point of the proof relies on Facts **F1–F4** stated in **Appendix A**. For any $y_d \in \mathcal{R}$, let $\bar{A}_d \triangleq \frac{\partial \bar{F}}{\partial z}(\varphi(y_d); y_d)$, $A_d \triangleq \frac{\partial F}{\partial x}(\varphi(y_d))$, $g_d \triangleq g(\psi(\phi^{-1}(y_d)))$, $\eta_d \triangleq \eta(\phi^{-1}(y_d))$, $\phi'_d \triangleq \phi'(\phi^{-1}(y_d))$, $H_d \triangleq \left[\frac{\partial h}{\partial x}\right]^T(\psi(\phi^{-1}(y_d)))$, and observe (from the definition of ϕ in Assumption A6) that $h(\psi(\phi^{-1}(y_d))) = y_d$. Thus, from Eq. (5), we have that

$$\bar{A}_d = \begin{pmatrix} A_d & g_d \\ -sk\eta_d H_d^T & 0 \end{pmatrix} \tag{6}$$

Since (according to Assumption A4 and point R4 of Remark 1), for any $y_d \in \mathcal{R}$, A_d is Hurwitz, then for any positive definite symmetric matrix $\bar{S}$ there exists $\bar{R} = \bar{R}^T > 0$ such that $\bar{R}A_d + A_d^T\bar{R} = -\bar{S}$ (see for instance [1, Theorem 4.6]). Let

$$\bar{P} = \begin{pmatrix} P & p \\ p^T & p_{n+1} \end{pmatrix} \tag{7}$$

and

$$\bar{Q} = \begin{pmatrix} Q & 0_n \\ 0_n^T & q_{n+1} \end{pmatrix} \tag{8}$$

where $P \in \mathbb{R}^{n\times n}$ is the positive definite symmetric solution of

$$PA_d + A_d^T P = -I_n \tag{9a}$$

$q_{n+1} \in \mathbb{R}$ is a constant satisfying

$$q_{n+1} > \max\left\{0\,,\, g_d^T\left[PA_d^{-1} + \left[A_d^{-1}\right]^T P\right]g_d\right\} \tag{9b}$$

$p_{n+1} \in \mathbb{R}$, $p \in \mathbb{R}^n$, and $Q \in \mathbb{R}^{n\times n}$ are defined as

$$p_{n+1} = \frac{\nu_d}{2k\eta_d|\phi'_d|} \tag{9c}$$

$$p = \left[A_d^{-1}\right]^T \left[\frac{\nu_d}{2\phi'_d} H_d - P g_d\right] \tag{9d}$$

$$Q = I_n + sk\eta_d R_d \tag{9e}$$

$\nu_d \in \mathbb{R}$ and $R_d \in \mathbb{R}^{n\times n}$ stand for

$$\nu_d = q_{n+1} - g_d^T \left[P A_d^{-1} + \left[A_d^{-1}\right]^T P\right] g_d \tag{9f}$$

$$R_d = pH_d^T + H_d p^T \tag{9g}$$

and the positive constant k is considered to satisfy

$$k < \min\left\{\frac{1}{\eta_d\|R_d\|}\, ,\, \frac{\nu_d}{2\eta_d|\phi'_d|p^T P^{-1} p}\right\} \tag{9h}$$

Notice, from Eqs. (9e) and (9g), that $Q^T = Q$. Note on the other hand that, from the satisfaction of inequality (9h), we have that $0 < 1 - k\eta_d\|R_d\| = \lambda_{\min}(I_n) - \|sk\eta_d R_d\|$, and consequently Q is a positive definite symmetric matrix. From this, (8), (9b), and Fact F3, we have that $\bar{Q}^T = \bar{Q} > 0$. Observe further that, from the satisfaction of inequality (9h), and Eq. (9c), (under the consideration of Fact F1) we have that $0 < \frac{\nu_d}{2k\eta_d|\phi'_d|} - p^T P^{-1} p = p_{n+1} - p^T P^{-1} p$. From this, the positive definite symmetric character of P, Eq. (7), and Fact F3, we see that $\bar{P}^T = \bar{P} > 0$ too. Furthermore, from Eqs. (6)–(8) and the expressions developed in Appendix B, one verifies that

$$\begin{aligned}
-\bar{Q} &= \begin{pmatrix} -Q & 0_n \\ 0_n^T & -q_{n+1} \end{pmatrix} \\
&= \begin{pmatrix} PA_d + A_d^T P - sk\eta_d\left[pH_d^T + H_d p^T\right] & Pg_d + A_d^T p - sk\eta_d p_{n+1} H_d \\ g_d^T P + p^T A_d - sk\eta_d p_{n+1} H_d^T & g_d^T p + p^T g_d \end{pmatrix} \\
&= \begin{pmatrix} PA_d - sk\eta_d p H_d^T & Pg_d \\ p^T A_d - sk\eta_d p_{n+1} H_d^T & p^T g_d \end{pmatrix} + \begin{pmatrix} A_d^T P - sk\eta_d H_d p^T & A_d^T p - sk\eta_d p_{n+1} H_d \\ g_d^T P & g_d^T p \end{pmatrix} \\
&= \begin{pmatrix} P & p \\ p^T & p_{n+1} \end{pmatrix} \begin{pmatrix} A_d & g_d \\ -sk\eta_d H_d^T & 0 \end{pmatrix} + \begin{pmatrix} A_d^T & -sk\eta_d H_d \\ g_d^T & 0 \end{pmatrix} \begin{pmatrix} P & p \\ p^T & p_{n+1} \end{pmatrix} \\
&= \bar{P}\bar{A}_d + \bar{A}_d^T \bar{P}
\end{aligned}$$

Hence, given the positive definite symmetric matrix $\bar{Q}$ in (8), $\bar{P}$ in (7) turns out to be the positive definite symmetric solution of the (Lyapunov) equation $\bar{P}\bar{A}_d + \bar{A}_d^T\bar{P} = -\bar{Q}$, and consequently (according to [1, Theorem 4.6]) $\bar{A}_d$ in (6) is a Hurwitz matrix. Note that this holds for any $y_d \in \mathcal{R}$.

P5. For any (constant) $y_d \in \mathcal{R}$, the next points follow.

(a) Since Assumption A5a is uniform in u on Υ, and θ remains in Υ (according to point P3 above), the eventual invariant sets on $\partial\Omega \times \Upsilon \subset \partial(\Omega\times\Upsilon)$, if any, cannot be attractive. On the other hand, through the consideration of η in (3), the auxiliary dynamics defines two additional equilibrium points on $\mathrm{int}(\Omega) \times \partial\Upsilon \subset \partial(\Omega\times\Upsilon)$: $\bar{\varphi} \triangleq (\psi^T(\bar{u}), \bar{u})^T$ and $\underline{\varphi} \triangleq (\psi^T(\underline{u}), \underline{u})^T$. Let $g_{\bar{u}} \triangleq g(\psi(\bar{u}))$, $g_{\underline{u}} \triangleq g(\psi(\underline{u}))$, $\eta'_{\bar{u}} \triangleq \eta'(\bar{u})$, and $\eta'_{\underline{u}} \triangleq \eta'(\underline{u})$; let us further define $\bar{A}_{\bar{u}} \triangleq \frac{\partial \bar{F}}{\partial z}\Big|_{z=\bar{\varphi}}$, $\bar{A}_{\underline{u}} \triangleq \frac{\partial \bar{F}}{\partial z}\Big|_{z=\underline{\varphi}}$, $A_{\bar{u}} \triangleq \frac{\partial F}{\partial x}\big|_{x=\psi(\bar{u})}$, $A_{\underline{u}} \triangleq \frac{\partial F}{\partial x}\big|_{x=\psi(\underline{u})}$, and let $\bar{P}_{\bar{u}}$, $\bar{P}_{\underline{u}}$, $P_{\bar{u}}$, $P_{\underline{u}}$ respectively denote the characteristic polynomials of these matrices. From Eq. (5) (and the definition of η in the statement of Proposition 1), we have that

$$\bar{A}_{\bar{u}} = \begin{pmatrix} A_{\bar{u}} & g_{\bar{u}} \\ 0_n^T & sk\eta'_{\bar{u}}(y_d - \phi(\bar{u})) \end{pmatrix}$$

and

$$\bar{A}_{\underline{u}} = \begin{pmatrix} A_{\underline{u}} & g_{\underline{u}} \\ 0_n^T & sk\eta'_{\underline{u}}(y_d - \phi(\underline{u})) \end{pmatrix}$$

whence we see that

$$\bar{P}_{\bar{u}}(\lambda) = (\lambda - sk\eta'_{\bar{u}}(y_d - \phi(\bar{u})))P_{\bar{u}}(\lambda)$$

and

$$\bar{P}_{\underline{u}}(\lambda) = (\lambda - sk\eta'_{\underline{u}}(y_d - \phi(\underline{u})))P_{\underline{u}}(\lambda)$$

Let us note, from the monotonic character of $\phi(u)$ (stated in Assumption A6), that $s = \mathrm{sign}(\phi(\bar{u}) - \phi(\underline{u})) = -\mathrm{sign}(y_d - \phi(\bar{u})) = \mathrm{sign}(y_d - \phi(\underline{u}))$. On the other hand, from the definition of η in the statement of Proposition 1, we have that $\eta'_{\bar{u}} < 0$ while $\eta'_{\underline{u}} > 0$. Therefore, $\bar{P}_{\bar{u}}(\lambda)$ and $\bar{P}_{\underline{u}}(\lambda)$ prove to have a root $\lambda_{\bar{u}} = -k\eta'_{\bar{u}}|y_d - \phi(\bar{u}))| > 0$ and $\lambda_{\underline{u}} = k\eta'_{\underline{u}}|y_d - \phi(\underline{u})| > 0$ respectively. Hence, both $\bar{A}_{\bar{u}}$ and $\bar{A}_{\underline{u}}$ have positive eigenvalues and consequently $\bar{\varphi}$ and $\underline{\varphi}$ are both unstable equilibrium points.

(b) Observe that the closed loop is composed of two subsystems evolving at different time scales. Indeed, the control parameter k in (3) has a direct effect on the auxiliary subsystem solution speed. Let us define the time variable $\tau = kt$. Expressing the closed-loop dynamics in this time scale, we have

$$\begin{aligned} k\frac{dx}{d\tau} &= F(x, \theta) \\ \frac{d\theta}{d\tau} &= s\eta(\theta)[y_d - h(x)] \triangleq G(x, \theta) \end{aligned} \tag{10}$$

In view of the adopted form, system (10) is analyzed as a standard singular perturbation model. Let us begin by noting that, in accordance to the standing assumptions, $F(x, \theta) = 0_n$ has a unique isolated root on Ω, namely $x = \psi(\theta)$. On the other hand, as highlighted in point R2 of Remark 1, for every *frozen* $\theta \in \mathrm{int}(\Upsilon)$, $x = \psi(\theta)$ is an exponentially stable equilibrium of the boundary-layer system $\frac{dx}{dt} = F(x, \theta)$ and, from point R3 of Remark 1, it is clear that $\mathrm{int}(\Omega)$ is a subset of the region of asymptotic stability of $x = \psi(\theta)$. Consider now the reduced system

$$\frac{d\theta}{d\tau} = s\eta(\theta)[y_d - \phi(\theta)] = G(\psi(\theta), \theta) \tag{11}$$

and let $\theta_d \triangleq \phi^{-1}(y_d)$. Noting that

$$G'\big(\psi(\theta), \theta\big)\Big|_{\theta=\theta_d} = -\eta(\theta_d)\big|\phi'(\theta_d)\big| < 0$$

(where Fact F4, stated in Appendix A, has been considered), we conclude that $\theta = \theta_d$ is an exponentially stable equilibrium of (11) [1, Theorem 4.15]. Moreover, from the monotonic character of ϕ and the definition of s, we have that $(\theta - \theta_d)G\big(\psi(\theta), \theta\big) < 0$, $\forall \theta \in \mathrm{int}(\Upsilon)$, and consequently $\mathrm{int}(\Upsilon)$ constitutes the region of asymptotic stability of $\theta = \theta_d$, denoted $R^A_{\theta_d}$, *i.e.* $R^A_{\theta_d} = \mathrm{int}(\Upsilon)$. Then, for any $\bar{\theta}(0) = \theta_0 \in \mathrm{int}(\Upsilon)$, $\bar{\theta}(t) \to \theta_d$ as $t \to \infty$, where $\bar{\theta}(t)$ denotes the solution of the reduced system (11). Moreover, according to [1, Theorem 4.17], there exists a Lyapunov function $V_{\theta_d}(\theta)$ defined on

$R^A_{\theta_d}$, that proves the asymptotic stability of $\theta = \theta_d$ and, given any $c > 0$, $\{V_{\theta_d}(x) \leq c\}$ is a compact subset of $R^A_{\theta_d}$. Since this holds for any positive value of c, and in view of the smoothness properties of f, g, ϕ, and η, we conclude from [1, Theorem 11.2] that there is a positive constant $\bar{k}$ such that for all $x_0 \in \mathrm{int}(\Omega)$, all $\theta_0 \in \mathrm{int}(\Upsilon)$, and any $k \in (0, \bar{k})$: $\theta(t) - \bar{\theta}(t) = O(k)$, $\forall t \geq 0$, and given any $t_b > 0$, there is $\bar{k}^* \leq \bar{k}$ such that, for any $k \in (0, \bar{k}^*)$: $x(t) - \psi(\bar{\theta}(t)) = O(k)$, $\forall t \geq t_b$. In other words, for a sufficiently small value of k, there exist a finite time $t_b > 0$ and a positive value k_1 such that, for any $z(0) = (x_0, \theta_0) \in \mathrm{int}(\Omega \times \Upsilon)$, $\|z(t) - (\psi(\bar{\theta}(t)), \bar{\theta}(t))\| \leq k_1 k$, $\forall t \geq t_b$, while $\bar{\theta}(t) \to \theta_d$ as $t \to \infty$. Hence, $z(t)$ turns out to approach $\varphi(y_d)$, or more precisely, $z(t)$ gets into a small region around $\varphi(y_d)$ where it will remain afterwards. Moreover, the smaller is the value of k, the closer $z(t)$ gets to $\varphi(y_d)$ (or the smaller is the neighborhood of $\varphi(y_d)$ where $z(t)$ enters to remain therein). Furthermore, from point P4 above, $\varphi(y_d)$ happens to be exponentially stable [1, Theorem 4.15]. Therefore, by choosing k small enough, $z(t)$ finishes up by getting into the region of exponential stability of $\varphi(y_d)$, and consequently $z(t) \to \varphi(y_d)$ as $t \to \infty$. Thus, a small enough value of k ensures the absence of invariant sets in $\mathrm{int}(\Omega \times \Upsilon)$ other than $\{\varphi(y_d)\}$.

P6. Observe that the function $\bar{\phi}(y_d) \triangleq \bar{h}(\varphi(y_d)) = h(\psi(\phi^{-1}(y_d))) = \phi(\phi^{-1}(y_d)) = y_d$ is (linearly) increasing on $\mathcal{R}$.

Points P1–P6, above, show that the closed-loop state model in Eqs. (4) has properties analogous to those of the open-loop system stated through Assumptions A1–A6, and consequently, points R1–R4 of Remark 1 analogously apply to the closed loop. Thus, from points P1, P3, and P5a, we have (in addition to existence and uniqueness of solutions) that, for any $z_0 \in \mathrm{int}(\Omega \times \Upsilon)$, $z(t; z_0) \in \mathrm{int}(\Omega \times \Upsilon)$, $\forall t \geq 0$. Furthermore, from points P2–P5, $\{\varphi(y_d)\}$ is concluded to be the unique invariant set in $\mathrm{int}(\Omega \times \Upsilon)$, and from the compact character of $\Omega \times \Upsilon$ (which implies boundedness of every trajectory remaining in $\Omega \times \Upsilon$), we have —according to [1, Lemma 4.1]— that, for any $z_0 \in \mathrm{int}(\Omega \times \Upsilon)$, $z(t; z_0) \to \varphi(y_d)$ as $t \to \infty$ (which was actually concluded in point P5b above). Thus (according to Eq. (2); see also point P6), we conclude that $y(t) \to y_d$ $[= h(\psi(\phi^{-1}(y_d)))]$ as $t \to \infty$, with $u(t) \in \mathrm{int}(\Upsilon)$, $\forall t \geq 0$, and $x(t) \in \mathrm{int}(\Omega)$, $\forall t \geq 0$. □

It is worth pointing out that in the general context considered in this work, where any output (error) variable may be involved in (3), the closed-loop structural stability does not necessarily hold for any (positive) control gain k. Stability of the *relocated* equilibrium point may be lost for high values of k giving rise to unexpected asymptotical behaviors (such as a sustained oscillation). Nevertheless, the small-enough condition on k (stated in Proposition 1) may be relaxed under additional requirements or for specific output variables as stated through the following corollaries that do not additionally require twice continuous differentiability of f and g in (1a).

Corollary 1

Under Assumptions A1–A6, consider the closed-loop system in Eqs. (4), where the control scheme (2)-(3) has been applied. If there is a continuously differentiable scalar function $V(z; y_d)$ such that $\dot{V}(z; y_d) = \frac{\partial V}{\partial z}\bar{F}(z; y_d) \leq 0, \forall (z; y_d) \in \Omega \times \Upsilon \times \mathcal{R}$, with $\dot{V}(\varphi(y_d); y_d) = 0, \forall y_d \in \mathcal{R}$, and $\dot{V}(z; y_d) < 0$, $\forall (z; y_d) \in \mathrm{int}(\Omega \times \Upsilon) \times \mathcal{R} \setminus \{(\varphi(y_d), y_d)\}$, uniformly in k (*i.e.* $\forall k > 0$), then the regulation objective is achieved, avoiding input saturation, for any control gain value $k > 0$.

The proof of Corollary 1 follows directly from La Salle's invariance principle, the unattractive nature of the invariant sets on $\partial(\Omega \times \Upsilon)$, and the uniformity in k of the stated conditions.

Corollary 2

Consider the system in Eqs. (1) under the satisfaction of Assumptions A1–A6. Assume there is a continuously differentiable scalar function $V(x)$, independent of u, such that $\dot{V}(x, u) = \frac{\partial V}{\partial x}F(x, u) \leq 0$, $\forall (x, u) \in \Omega \times \Upsilon$, with $\dot{V}(\psi(u), u) = 0$, $\forall u \in \Upsilon$, and $\dot{V}(x, u) < 0$, $\forall (x, u) \in \mathrm{int}(\Omega) \times \Upsilon \setminus \{(\psi(u), u)\}$, and suppose that $h(x) = s\frac{\partial V}{\partial x}g(x) + y_d$ (with s as defined in the statement of Proposition 1, *i.e.* $s = \mathrm{sign}\big(\phi(\bar{u}) - \phi(\underline{u})\big)$). Then, by applying the control design methodology of Proposition 1, the regulation objective is achieved, avoiding input saturation, for any control gain value $k > 0$.

Proof

Consider the closed-loop system (1)–(3), where the control design methodology of Proposition 1 has been applied. Recall from the proof of Proposition 1 that $\Omega \times \Upsilon$ is a positively invariant compact set in the state space of the closed loop. Let us define

$$V_c(x, \theta) = V(x) + W(\theta)$$

with

$$W(\theta) = \int_{u_d}^{\theta} \varpi(\vartheta) d\vartheta$$

where $u_d \triangleq \phi^{-1}(y_d)$ and

$$\varpi(\theta) \triangleq \frac{\theta - u_d}{k\eta(\theta)}$$

Observe that $(\theta - u_d)\varpi(\theta) > 0$, $\forall \theta \in \Upsilon \setminus \{u_d\}$. Hence, $W(\theta)$ is positive definite on Υ with respect to u_d, *i.e.* $W(\theta) \geq 0$, $\forall \theta \in \Upsilon$, with $W(\theta) = 0 \iff \theta = u_d$ (see for instance [1, Example 4.2]). Observe further that, in view of the properties of η, $W(\theta) \to \infty$ as $\theta \to \partial\Upsilon$ (*i.e.* as $\theta \to \underline{u}^+$ or $\theta \to \bar{u}^-$). Let us consider the derivative of V_c along the closed loop trajectories, which is given by

$$\begin{aligned} \dot{V}_c(x, \theta) &= \frac{\partial V}{\partial x} F(x, \theta) + \varpi(\theta)\dot{\theta} \\ &= \frac{\partial V}{\partial x}[f(x) + g(x)\theta] + s(y_d - h(x))(\theta - u_d) \\ &= \frac{\partial V}{\partial x}[f(x) + g(x)u_d] + \left[\frac{\partial V}{\partial x} g(x) + s(y_d - h(x))\right](\theta - u_d) \end{aligned}$$

Since $h(x) = s\frac{\partial V}{\partial x} g(x) + y_d$, we get

$$\dot{V}_c(x, \theta) = \frac{\partial V}{\partial x}[f(x) + g(x)u_d] = \dot{V}(x, u_d)$$

From the considered assumptions, we see that $\dot{V}_c(x, \theta) \leq 0$, $\forall (x, \theta) \in \Omega \times \Upsilon$, with $\dot{V}_c(x, \theta) = 0$ on $E = \{(x, \theta) \in \Omega \times \Upsilon : x = \psi(u_d)\}$. Then, according to La Salle's invariance principle, every trajectory with initial condition in $\Omega \times \Upsilon$ approaches the largest invariant in E asymptotically in time. Since E is a 1-dimensional manifold, no limit cycles may exist in E. Hence, since $\varphi(y_d)$ is the unique equilibrium in $\text{int}(\Omega \times \Upsilon)$, actually located in $\text{int}(E)$, and considering the unstable character of the equilibrium points on ∂E, we conclude that, for any $(x, \theta)(0) \in \text{int}(\Omega \times \Upsilon)$, $(x, \theta)(t) \to \varphi(y_d)$ as $t \to \infty$, with $(x, \theta)(t) \in \text{int}(\Omega \times \Upsilon)$, $\forall t \geq 0$. Since the proof does not depend on the value of k, the output regulation objective, avoiding input saturation, is guaranteed whatever positive value is assigned to k. □

4. APPLICATIONS

The results in [8] and [11] show that double-pipe heat exchangers belong to the type of systems that may be regulated through the approach developed in this work. In particular, such references prove that the control design methodology in Proposition 1 may be applied —with $s = -1$ in (3)— to the heat exchanger model considered therein for the regulation of the hot fluid outlet temperature —by means of the cold fluid flow rate— avoiding input saturation. Its implementation on a bench-scale pilot heat exchanger, was developed —taking $\eta(\theta) = (\theta - F_{cl})(F_{cu} - \theta)$— through experimental and simulation tests. Successful results were obtained, which are shown in [11]. It is worth pointing out that, in this case, the asymptotic stability of the unique closed-loop equilibrium does not hold for any value of the control gain k (see Eq. (3)), as may be corroborated through the proof of the main result in [11]. Simulation results (not presented in [11]) obtained from implementations of the model considered in [11] in closed loop with the proposed scheme have shown that by fixing a sufficiently high value of k in (3), a sustained oscillatory behavior of the output and input variables takes place. This corroborates that the sufficiently-small restriction on the control gain is not just a consequence of the developed closed-loop analysis but is actually a genuine condition to keep the required stability properties for some systems whose closed-loop structural stability is lost at sufficiently high values of the control gain.

In the rest of this section, two additional cases, where the proposed approach may be applied, are presented; namely: bioreactors and binary distillation columns.

4.1. Bioreactors

Under basic assumptions —mainly, equal feed and effluent flow rates (constant volume reactor)—, a simple but suitable state-space representation of bioreactors that globally describes the dynamic behavior of a large class of such processes [9] is given by

$$\dot{X} = [\mu(S) - D]X \tag{12a}$$

$$\dot{S} = -\frac{\mu(S)X}{Y} + D(S_{in} - S) \tag{12b}$$

where X and S are the biomass and substrate concentrations respectively —both nonnegative variables—; Y is the yield coefficient of the substrate consumption by the biomass, S_{in} is the substrate concentration in the feed stream —both (Y and S_{in}) considered (positive) constant values—; D is the dilution rate, and $\mu(S)$ is the specific microbial growth rate, defined for our purposes —as in [9]— by the Haldane law:

$$\mu(S) = \frac{\mu_m S}{K_S + S + S^2/K_I} \tag{13}$$

where μ_m is the maximum specific biomass growth rate, K_S and K_I are the saturation and substrate inhibition parameters, respectively, all three being positive constants. The dilution rate D can be made vary between a lower limit value $\underline{D} > 0$ and an upper bound $\bar{D} < \mu(S_{in})$, *i.e.* such that

$$D \in \Upsilon = [\underline{D}, \bar{D}] \subset (0, \mu(S_{in})) \tag{14}$$

In fact, D is considered the input variable by means of which regulation of the substrate concentration S is to be achieved. Note that by defining $x = (X, S)^T$, $u = D$, and $y = S$, the bioreactor dynamics in Eqs. (12) adopts the form of the state model in Eqs. (1), *i.e.* $\dot{x} = f(x) + g(x)u \triangleq F(x, u)$, with

$$f(x) = \begin{pmatrix} \mu(x_2)x_1 \\ -\frac{\mu(x_2)x_1}{Y} \end{pmatrix} \quad , \quad g(x) = \begin{pmatrix} -x_1 \\ S_{in} - x_2 \end{pmatrix} \tag{15a}$$

and

$$h(x) = x_2 \tag{15b}$$

Further, letting[‡] $\Omega_0 = \{(x_1, x_2) \in \mathbb{R}^2_+ : x_2 \leq S_{in}\}$ and defining $\Omega = \Omega_1 \cap \Omega_2$ with, for $i = 1, 2$:

$$\Omega_i \triangleq \{(x_1, x_2) \in \Omega_0 : \omega_i(x_1, x_2) \leq 0\}$$

where

$$\omega_i(x_1, x_2) = (-1)^i \left(\frac{x_1}{Y} + x_2 - c_i\right)$$

[‡] Ω_0 may be considered a physically coherent state-space domain for system (12). This is due to the nonnegative character of the state variables and the substrate consumption by the biomass. Such consumption reduces the substrate concentration in the reactor, rendering it lower than that in the feed stream.

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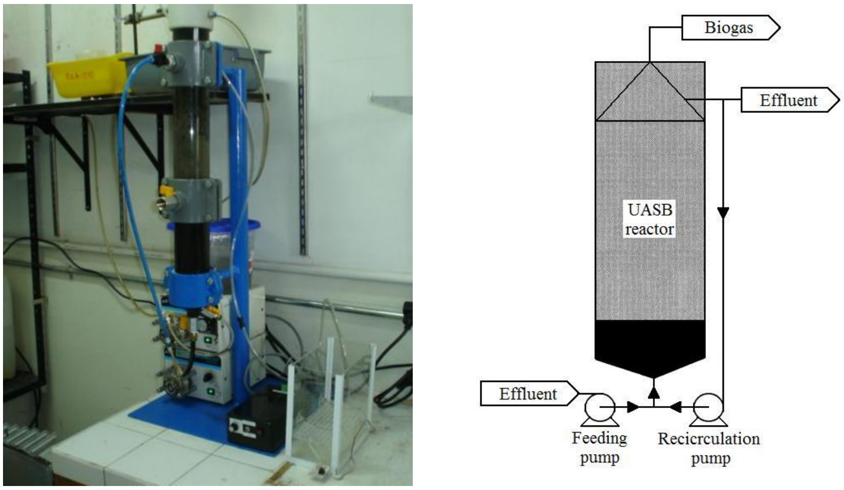

Figure 1. The UASB bioreactor experimental setup and its schematic diagram

c_1 and c_2 being (any) constants satisfying

$$0 \leq c_1 < S_{in} < c_2 \tag{16}$$

the considered bioreactor model satisfies Assumptions A1–A6 —with $\phi'(u) > 0$, $\forall u \in \Upsilon$— as thoroughly proven in Appendix C. Therefore, the control design methodology in Proposition 1 may be applied —with $s = 1$ in (3)— to the bioreactor model for the regulation of substrate concentration S —by means of the dilution rate D— avoiding input saturation. As a matter of fact, it is further proven in Appendix C that the proposed methodology can be applied in this case with any $k > 0$.

Experimental results

The proposed methodology was applied for substrate concentration regulation on a laboratory-scale Up-flow Anaerobic Sludge Blanket (UASB) reactor of 2.1 litters; see Fig. 1. The reactor was fed with industrial wastewater from a brewery, which was diluted and conditioned to ensure an experimental influent concentration $S_{in} = 3$ g/l (the reactor temperature and pH were respectively fixed at 35 °C and 7). A detailed description of this reactor, including its biochemical properties and technical aspects of the setup, can be found in [14].

With dilution rate lower- and upper-bound values of $\underline{D} = 0.0476$ d^{-1} and $\bar{D} = 1$ d^{-1} in the setup (time units are expressed in days), closed-loop experimental tests were carried out taking $\eta(\theta) = \sin\left(\pi(\theta - \underline{D})/(\bar{D} - \underline{D})\right)$ in the auxiliary dynamics of the control law (*i.e* in (3)). The control gain and desired output (substrate concentration) values were fixed at $k = 0.9$ l/(g·d^2) and $y_d = 0.5$ g/l. At $t = 0$, when the loop was closed, the reactor substrate concentration had a steady-state initial value of $S(0) = 1.5$ g/l, while the controller auxiliary state was assigned an initial value of $\theta(0) = 0.4762$ d^{-1}. The closed-loop performance was further tested against a (parameter) perturbation carried out by suddenly changing the influent concentration from $S_{in} = 3$ g/l to $S_{in} = 2$ g/l at $t = 20$ d.

For comparison purposes, closed-loop tests with a (conventional) PI controller, *i.e.* $u(t) = k_p[y_d - y(t)] + k_i \int_0^t (y_d - y(\tau))d\tau$, were also carried out. The previous experimental conditions were reproduced for this controller. The control gains were fixed at $k_p = 2$ l/(g·d) and $k_i = 0.75$ l/(g·d^2) (tuned essentially following the Ziegler-Nichols method).

The closed-loop output responses and input signals that resulted from the experiments with the tested algorithms are shown in Fig. 2. Observe that the controller resulting from the application of the proposed methodology achieved the regulation objective, and successfully recovered from the parameter sudden perturbation, avoiding input saturation throughout the whole experimental

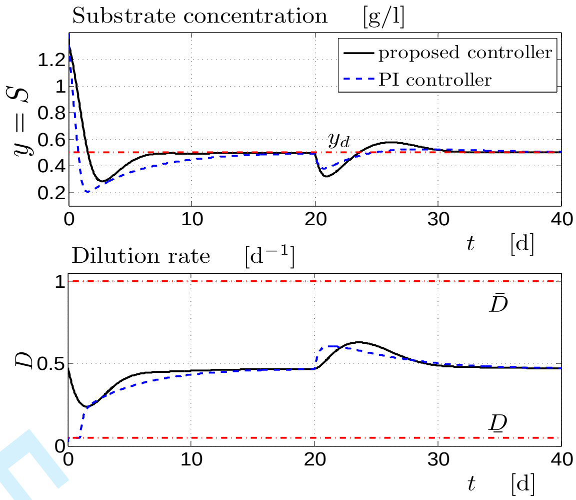

Figure 2. UASB bioreactor closed-loop tests: experimental results

test. The desired convergence and recovery took place with the PI controller too (although no real improvement on the closed-loop performance can be appreciated). In this case, however, input saturation is observed during an initial time interval. This is not necessarily disadvantageous for the process, but it could be inconvenient for the actuators. Besides, in a general context, such a phenomenon may give rise to undesirable effects as experimentally corroborated in [11] in the case of heat exchangers. Let us further note that the sudden reactions of the PI controller are not necessarily beneficial to the closed-loop performance; for instance, shorter stabilization times were not observed from the experimental tests. On the contrary, appealing results in this direction were appreciated through the proposed controller whose smoother reactions seem to better suit to the bioreactor dynamics.

4.2. Binary distillation columns

Under standard assumptions, an n-tray binary distillation column (with a saturated liquid being fed through the feedstream) may be suitably modeled as [10]

$$H_1\dot{x}_1 = V\kappa(x_2) - Vx_1 \tag{17a}$$

$$H_j\dot{x}_j = Lx_{j-1} + V\kappa(x_{j+1}) - Lx_j - V\kappa(x_j)\,, \quad j = 2,\dots,j_f-1 \tag{17b}$$

$$H_{j_f}\dot{x}_{j_f} = Lx_{j_f-1} + V\kappa(x_{j_f+1}) - (L+F_f)x_{j_f} - V\kappa(x_{j_f}) + F_f z_f \tag{17c}$$

$$H_j\dot{x}_j = (L+F_f)x_{j-1} + V\kappa(x_{j+1}) - (L+F_f)x_j - V\kappa(x_j)\,, \quad j = j_f+1,\dots,n-1 \tag{17d}$$

$$H_n\dot{x}_n = (L+F_f)x_{n-1} - (L+F_f-V)x_n - V\kappa(x_n) \tag{17e}$$

where $j \in \{1,\dots,n\}$ denotes the tray index, with $j=1$ corresponding to the reflux drum, $j = j_f \in \{2,\dots,n-1\}$ to the feed tray, and $j = n \geq 3$ to the bottom;§ L and V are respectively the reflux (liquid) and reboiler (vapor) molar flowrates; F_f is the feed molar flowrate and $z_f \in (0,1)$ is the feed composition (molar fraction); for every $j = 1,\dots,n$: $H_j > 0$ represents the (molar) liquid holdup, considered constant, x_j the liquid molar fraction, and $\kappa(x_j)$ is the vapor molar fraction, commonly modeled as [15, §M10.1]

$$\kappa(\varsigma) = \frac{\alpha\varsigma}{1+(\alpha-1)\varsigma} \tag{18}$$

§Let us note that Eqs. (17b) take place if $j_f > 2$ and Eqs. (17d) arise if $j_f < n-1$, while none of these sets of equations takes place if $n = 3$.

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where $\alpha > 1$, considered constant, is the relative volatility.

Remark 3
From (18), we have that

$$\kappa'(\varsigma) = \frac{\alpha}{[1 + (\alpha - 1)\varsigma]^2} \tag{19}$$

wherefrom one corroborates that $\kappa(\varsigma)$ is a continuously differentiable function on $\mathbb{R}_+$ such that $\kappa'(\varsigma) \in (0, \alpha]$, $\forall \varsigma \geq [0, \infty)$, and in particular $\kappa'(\varsigma) \in \left[\frac{1}{\alpha}, \alpha\right]$, $\forall \varsigma \in [0, 1]$. Moreover, from (19), we get $\kappa''(\varsigma) = -2\alpha(\alpha - 1)/[1 + (\alpha - 1)\varsigma]^3$, which is a well-defined expression for all $\varsigma \geq 0$.

In our analytical setting, F_f, z_f, and V are considered constant (and are by nature positive), while L may be varied between a lower bound $\underline{L} > \max\{0, V - F_f\}$ and an upper limit $\bar{L} < V$, such that¶

$$L \in \Upsilon = [\underline{L}, \bar{L}] \subset (\max\{0, V - F_f\}\, , V) \tag{20}$$

Actually, we consider L as the input variable through which regulation of x_1 is to be achieved.

Continuous-differentiability with respect to the system states at any combination of nonnegative values of these variables, and linearity in L, of the right-hand side expressions in Eqs. (17) are easily verifiable. Hence, by defining $x = (x_1, \ldots, x_n)^T$, $u = L$, and $y = x_1$, one straightforwardly corroborates that the system dynamics may be written in the state-model form of Eqs. (1), with vector functions f, g, and h satisfying Assumption A1 on $\mathbb{R}^n_+$. Moreover, from Remark 3, one can easily see that f and g are twice continuously differentiable. Furthermore, letting $\Omega \in \mathcal{I}^n$, with $\mathcal{I} \triangleq [0, 1]$, Assumptions A2–A5 are proven to be satisfied in [10].‖ In particular, the unique equilibrium point ψ is shown to satisfy

$$\psi(u) \in \{x \in \mathrm{int}(\Omega) : x_j > x_{j+1},\ \forall j = 1, \ldots, n - 1\} \quad \forall u \in \Upsilon \tag{21}$$

Further, by considering the function $\phi(u) = h(\psi(u))$, it turns out that $\phi'(u) > 0$, $\forall u \in \Upsilon$, as corroborated in Appendix D. Consequently, Assumption A6 is satisfied too with a strictly increasing $\phi(u)$ on Υ. Thus, the control design methodology in Proposition 1 may be applied —with $s = 1$ in (3)— to the distillation column model for the regulation of the distillate product molar fraction x_1 —by means of the reflux molar flowrate L— avoiding input saturation.

Simulation results

Considering the dynamical model of a 13-tray binary distillation column, with $j_f = 7$, closed-loop tests were carried out through numerical simulations. The considered parameter values were $V = 3.206$ mol/min, $F_f = 1$ mol/min, $z_f = 0.5$ mole fraction of light component, $\alpha = 1.5$, $H_1 = H_{13} = 5$ mol, $H_i = 0.5$ mol, $i = 2, \ldots, 12$ (these parameter values were taken from [15, §M10.4 & §M10.5]). The minimum and maximum reflux molar flowrate values were taken as $\underline{L} = 2.4$ mol/min and $\bar{L} = 3.2$ mol/min. The auxiliary dynamics in (3) was implemented using $\eta(\theta) = \mathrm{sech}\big(s - (\bar{L} + \underline{L})/2\big) - \mathrm{sech}\big((\bar{L} - \underline{L})/2\big)$, and a control gain $k = 0.52$ mol/min^2. The desired output value was defined as $y_d = 0.9$. The controller auxiliary state was assigned an initial value of $\theta(0) = 2.5$ mol/min. In the distillation column model, initial state conditions corresponding to a constant input value of $L = \theta(0) = 2.5$ mol/min (*i.e.* $L(t) = 2.5$, $\forall t \leq 0$) were considered (in particular $x_1(0) = 0.6814$). The closed-loop performance was further tested against a (parameter) perturbation carried out by suddenly changing the feed molar flowrate from $F_f = 1$ mol/min to $F_f = 0.812$ mol/min at $t = 150$ min.

For comparison purposes, an output feedback linearization (OFL) [7] control scheme was numerically implemented too. A 2nd order linear dynamics was aimed to be imposed to the output

¶Let us note that, through the consideration of (20), Assumption 1 in [10] is satisfied. Actually the consideration of (20) proves to be coherent since $V - L = D > 0$ and $L + F_f - V = B > 0$ are respectively the distillate and bottom product molar flowrates (respectively related to the light and heavy component outflows; see for instance [15, §M10.3]).

‖The dependence on —and continuous-differentiability with respect to— u of the unique equilibrium point, $\psi \in \mathrm{int}(\Omega)$, is a direct consequence of the continuous-differentiability of the right-hand side expressions of Eqs. (17) with respect to L; see for instance [1, §3.3].

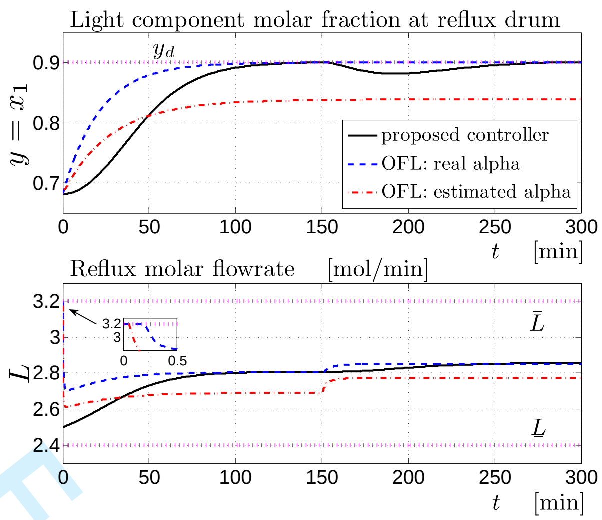

Figure 3. Distillation column closed-loop tests: simulation results

variable (the system state model has a relative degree equal to 2), *i.e.* $\ddot{y} = -k_1\dot{y} - k_2(y - y_d)$, with $k_1 > 0$ and $k_2 > 0$. The resulting controller thus designed is

$$u = \frac{1}{x_1 - x_2}\left[V[\kappa(x_2) - \kappa(x_3)] + \frac{\frac{V}{H_1}\left(\frac{V}{H_1} - k_1\right)[\kappa(x_2) - x_1] - k_2(x_1 - y_d)}{\frac{V}{H_1 H_2}\kappa'(x_2)}\right]$$

Compare the complexity of this expression with the simplicity of the regulator in Eqs. (2)–(3). Observe further that this expression does not only involve the output variable but also other state variables and several system parameters. The control gains were tuned as $k_1 = 30$ min^{-1} and $k_2 = 1.4$ min^{-2}. The previous simulation conditions were reproduced for this controller. First, closed-loop tests were performed using the exact values of all the system parameters in the control law. Then, tests were run taking in the control expression an estimated relative volatility value of $\hat{\alpha} = 1.55$ (while keeping the rest of the parameters at the exact values).

The closed-loop output responses and input signals that resulted from the simulations with the tested algorithms are shown in Fig. 3. Note that the algorithm resulting through the proposed methodology achieved the regulation objective, and successfully recovered from the parameter sudden perturbation, avoiding input saturation throughout the whole experimental test. Notice further that this was achieved through smooth control signals. On the contrary, sudden reactions were appreciated through the OFL schemes undergoing saturation during an initial (brief) time interval. Observe further that although the desired convergence and recovery are achieved (even improving the closed-loop performance) through the OFL algorithm in the ideal case where the real values of all the system parameters are involved, a steady-state error arises in the more realistic case where an inexact estimated value of the relative volatility was taken, in view of which the regulation objective could not even be achieved.

5. CONCLUSIONS

In this work, a generalized control scheme for the output feedback regulation of a special type of SISO systems with bounded input has been proposed. It gives rise to a simple dynamic controller that guarantees the regulation objective, avoiding input saturation, without requiring any additional system information (apart from the output variable), and for any state initial condition within a specific set where the system is known to satisfy the requested characterization (and which could comprehend the whole natural state space of the considered plant). The type of systems that may be regulated through the proposed scheme has been thoroughly characterized. Double-pipe heat exchangers, bioreactors, and binary distillation columns have been shown to be part of such set of processes. Furthermore, it has been possible to corroborate the efficiency of the proposed methodology through experimental implementations on laboratory-scale processes such as a UASB bioreactor and a double-pipe heat exchanger, as well as through other simulations tests, which have shown successful results. Extensions of the proposed scheme for its application to some type of MIMO systems is considered by the authors a potential subject of future research. In this direction, some interesting results have already been presented for continuous stirred tank reactors in [12].

A. FACTS

F1. Let $B \in \mathbb{R}^{n\times n}$. If $B^T = B > 0$, then B is non-singular and its inverse B^{-1} is a positive definite symmetric matrix (see for instance [16, p. 180]).

F2. Let $G = \begin{pmatrix} B & C \\ D & E \end{pmatrix}$, where $B \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{n\times m}$, $D \in \mathbb{R}^{m\times n}$, and $E \in \mathbb{R}^{m\times m}$, with B being non-singular. Then $\det(G) = \det(B)\cdot\det(E - DB^{-1}C)$ (see for instance [16, p. 46]).

F3. Let $G = \begin{pmatrix} B & C \\ D & E \end{pmatrix}$, where $B \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{n\times 1}$, $D \in \mathbb{R}^{1\times n}$, and $E \in \mathbb{R}$, with $B^T = B > 0$. Then G is a positive definite symmetric matrix if and only if $E - DB^{-1}C > 0$.**

F4. For any $u \in \Upsilon$:

$$\phi'(u) = -\frac{\partial h}{\partial x}(\psi(u))\left[\frac{\partial F}{\partial x}(\psi(u), u)\right]^{-1} g(\psi(u)) \tag{22}$$

and $|\phi'(u)| = s\phi'(u)$.

Proof of Fact F4

From Eqs. (1) and Assumptions A1, A2, A4, and A6, we have that, for any $u \in \Upsilon$:

$$\phi'(u) = \frac{\partial h}{\partial x}(\psi(u))\frac{d\psi}{du}(u) \tag{23}$$

and $\frac{d}{du}F(\psi(u), u) = 0_n$ *i.e.*

$$\begin{aligned}\frac{d}{du}F(\psi(u), u) &= \frac{\partial F}{\partial x}(\psi(u), u)\frac{d\psi}{du}(u) + \frac{\partial F}{\partial u}(\psi(u), u)\\ &= \frac{\partial F}{\partial x}(\psi(u), u)\frac{d\psi}{du}(u) + g(\psi(u)) = 0_n\end{aligned}$$

From this, we get (recall that $\frac{\partial F}{\partial x}(\psi(u), u)$ is non-singular in view of its Hurwitz character on Υ):

$$\frac{d\psi}{du}(u) = -\left[\frac{\partial F}{\partial x}(\psi(u), u)\right]^{-1} g(\psi(u)) \tag{24}$$

By substituting (24) into (23), Eq. (22) is obtained. Furthermore, notice that, in view of the monotonic character of $\phi(u)$ (stated in Assumption A6), $s = \mathrm{sign}\big(\phi(\bar{u}) - \phi(\underline{u})\big) = \mathrm{sign}(\phi'(u))$, $\forall u \in \Upsilon$, whence it is clear that $|\phi'(u)| = s\phi'(u)$. □

**This is a consequence of Fact F2 and the leading principal minor criterion for positive definite symmetric matrices (see for instance [1, p. 117] or [16, §8.5, Theorem 2]).

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B. DEVELOPMENTS OF THE PROOF OF PROPOSITION 1

From expressions (6)–(9) and Fact F4, the following developments are verified:

$$\begin{aligned}
-q_{n+1} &= -\nu_d - g_d^T\left[PA_d^{-1} + \left[A_d^{-1}\right]^T P\right]g_d \\
&= \frac{\nu_d}{2\phi'_d}\left[g_d^T\left[A_d^{-1}\right]^T H_d + H_d^T A_d^{-1} g_d\right] - g_d^T\left[PA_d^{-1} + \left[A_d^{-1}\right]^T P\right]g_d \\
&= g_d^T\left[A_d^{-1}\right]^T\left[\frac{\nu_d}{2\phi'_d}H_d - Pg_d\right] + \left[\frac{\nu_d}{2\phi'_d}H_d^T - g_d^T P\right]A_d^{-1}g_d \\
&= g_d^T p + p^T g_d
\end{aligned}$$

$$\begin{aligned}
0_n &= Pg_d - A_d^T\left[A_d^{-1}\right]^T Pg_d + A_d^T\left[A_d^{-1}\right]^T \frac{\nu_d}{2\phi'_d}H_d - \frac{\nu_d}{2\phi'_d}H_d \\
&= Pg_d + A_d^T\left[A_d^{-1}\right]^T\left[\frac{\nu_d}{2\phi'_d}H_d - Pg_d\right] - sk\eta_d\frac{\nu_d}{2k\eta_d|\phi'_d|}H_d \\
&= Pg_d + A_d^T p - sk\eta_d p_{n+1} H_d
\end{aligned}$$

and

$$-Q = -I_n - sk\eta_d R_d = PA_d + A_d^T P - sk\eta_d\left[pH_d^T + H_d p^T\right]$$

C. OPEN-LOOP ANALYSIS OF THE BIOREACTOR MODEL

Remark 4

Observe from (13) that $\lim_{S\to\infty}\mu(S) = \mu(0) = 0$. On the other hand, from (13), one gets

$$\mu'(S) = \frac{\mu_m(K_S - S^2/K_I)}{(K_S + S + S^2/K_I)^2} \tag{25}$$

whence one sees that $\mu(S)$ is differentiable at all $S \geq 0$, increasing on $[0, \sqrt{K_S K_I})$, decreasing on $(\sqrt{K_S K_I}, \infty)$, and has a maximum point at $S = \sqrt{K_S K_I} \triangleq S_M$, with $\mu(S_M) = \frac{\mu_m}{1+2\sqrt{K_S/K_I}} \triangleq \mu_M$ —such that $\mu(S) \leq \mu_M$, $\forall S \geq 0$, with $\mu(S) = \mu_M \iff S = S_M$— and thus $\mu(S) > 0$, $\forall S \in (0, \infty)$. Furthermore, denoting $\mathcal{D}(S)$ the denominator of $\mu(S)$ in (13), one sees that, since $\mathcal{D}(S) \geq K_S > 0$, $\forall S \geq 0$, higher order derivatives of $\mu(S)$ are well defined for all nonnegative values of S; in particular, denoting $\mathcal{N}(s)$ the numerator of $\mu'(S)$ in (25), $\mu''(S) = \left[\mathcal{D}(S)\mathcal{N}'(S) - 2\mathcal{N}(S)\mathcal{D}'(S)\right]/\mathcal{D}^3(S)$, whence it is clear that $\mu''(S)$ is well defined for all $S \geq 0$.

Observe, from expressions (15) and Remark 4, that f, g, and h are continuously differentiable on $\mathbb{R}_+^2$, and consequently Assumption A1 is satisfied. Moreover, from the differentiability properties of $\mu(S)$ discussed in Remark 4, f and g are concluded to be twice continuously differentiable.

Remark 5

By analyzing the equation $F(x,u) = 0_2$, three solutions are obtained; *i.e.* there exist x_∂^*, x_-^*, and x_+^* such that $F(x_\partial^*, u) = F(x_-^*, u) = F(x_+^*, u) = 0_2$, $\forall u \in \Upsilon$. More precisely

$$x_\partial^* = \begin{pmatrix} 0 \\ S_{in} \end{pmatrix}, \quad x_-^* = \begin{pmatrix} Y(S_{in} - S_-^*) \\ S_-^* \end{pmatrix}, \quad \text{and} \quad x_+^* = \begin{pmatrix} Y(S_{in} - S_+^*) \\ S_+^* \end{pmatrix}$$

with

$$S_\pm^* = \frac{K_I}{2}\left(\frac{\mu_m}{u} - 1\right) \pm \sqrt{\left[\frac{K_I}{2}\left(\frac{\mu_m}{u} - 1\right)\right]^2 - K_I K_S}$$

while further analysis shows that

$$S_-^* < \sqrt{K_S K_I} = S_M < S_+^* \tag{26}$$

$$S_+^* > S_{in} > S_-^* \tag{27}$$

and

$$x_\partial^* \in \partial\Omega_0 \quad , \quad x_-^* \in \mathrm{int}(\Omega_0) \quad , \quad x_+^* \neq \mathbb{R}_+^2 \tag{28}$$

$\forall u \in \Upsilon$ (observe that x_+^* does not have any physical sense). Let us note that if $\bar{D}$ had a value within $(\mu(S_{in}), \mu_M)$ and $S_{in} > S_M$ (recall Remark 4), then $x_+^* \in \mathrm{int}(\Omega_0)$. Nevertheless, as pointed out in [9], it is not this case —but it is rather that expressed in (28)— that is often encountered in applications.

From Remark 5, one sees that Assumption A2 is satisfied with $\psi(u) = x_-^*$, *i.e.*

$$\psi(u) = \begin{pmatrix} Y(S_{in} - \phi(u)) \\ \phi(u) \end{pmatrix} \tag{29}$$

where $\phi(u) = S_-^*$, *i.e.*

$$\phi(u) = \frac{K_I}{2}\left(\frac{\mu_m}{u} - 1\right) - \sqrt{\left[\frac{K_I}{2}\left(\frac{\mu_m}{u} - 1\right)\right]^2 - K_I K_S} \tag{30}$$

Remark 6
From (30), one gets

$$\phi'(u) = \frac{K_I \mu_m}{2u^2}\left[\frac{\frac{K_I}{2}\left(\frac{\mu_m}{u} - 1\right)}{\sqrt{\left[\frac{K_I}{2}\left(\frac{\mu_m}{u} - 1\right)\right]^2 - K_I K_S}} - 1\right]$$

whence one corroborates the differentiability of $\phi(u)$ —and consequently that of $\psi(u)$— on Υ. One also sees from this expression that $\phi'(u) > 0$, $\forall u \in \Upsilon$, showing that $\phi(u)$ is strictly increasing on Υ.

Observe that:

- $F_1(0, x_2, u) = 0$, $\forall (x_2, u) \in [0, S_{in}] \times \Upsilon$, while $F_2(0, x_2, u) = u(S_{in} - x_2) > 0$, $\forall (x_2, u) \in [0, S_{in}) \times \Upsilon$;
- $F_2(x_1, 0, u) = uS_{in} > 0$, $\forall (x_1, u) \in [0, \infty) \times \Upsilon$;
- $F_2(x_1, S_{in}, u) = -\frac{\mu(S_{in})x_1}{Y} < 0$, $\forall (x_1, u) \in (0, \infty) \times \Upsilon$;
- for each $i = 1, 2$: $\left[\left(\frac{\partial \omega_i}{\partial x} F\right)(x, u)\right]_{\omega_i(x)=0} = (-1)^i u(S_{in} - c_i) < 0$, $\forall (x, u) \in \{\Omega_i \times \Upsilon : \omega_i(x) = 0\}$.

This shows that, whatever value u takes in Υ, there is no point on $\partial\Omega$ where $F(x, u)$ points outwards; see Fig. 4. Then, for any $x_0 \in \Omega$, the resulting solution of the system state-space equation, $x(t; x_0)$, cannot leave Ω, *i.e.* $x(t; x_0) \in \Omega$, $\forall t \geq 0$. Therefore, Ω is a compact set that proves to be positively invariant with respect to the system dynamics uniformly in u on Υ. Further, simple developments show that $\max\{0, Y(c_1 - \phi(u))\} < \psi_1(u) = Y(S_{in} - \phi(u)) < Y(c_2 - \phi(u))$ and $\max\{0, c_1 - S_{in} + \phi(u)\} < \psi_2(u) = \phi(u) < \min\{S_{in}, c_2 - S_{in} + \phi(u)\}$, $\forall u \in \Upsilon$, *i.e.* $\psi(\Upsilon) \subset \Omega$; see Fig. 4. Thus, Assumption A3 is satisfied. Let us note that c_1 and c_2 may adopt suitable values (satisfying inequality (16)) such that any initial condition in Ω_0 be within Ω.

Let us consider the Jacobian matrix of $F(x, u)$, $\frac{\partial F}{\partial x}(x, u)$, *i.e.*

$$\frac{\partial F}{\partial x}(x, u) = \begin{pmatrix} \mu(x_2) - u & x_1 \mu'(x_2) \\ -\frac{\mu(x_2)}{Y} & -\frac{x_1 \mu'(x_2)}{Y} - u \end{pmatrix} \tag{31}$$

and define $A_\psi(u) \triangleq \frac{\partial F}{\partial x}(\psi(u), u)$. A direct substitution into (31) yields

$$A_\psi(u) = \begin{pmatrix} 0 & Y[S_{in} - \phi(u)]\mu'(\phi(u)) \\ -\frac{u}{Y} & -[S_{in} - \phi(u)]\mu'(\phi(u)) - u \end{pmatrix}$$

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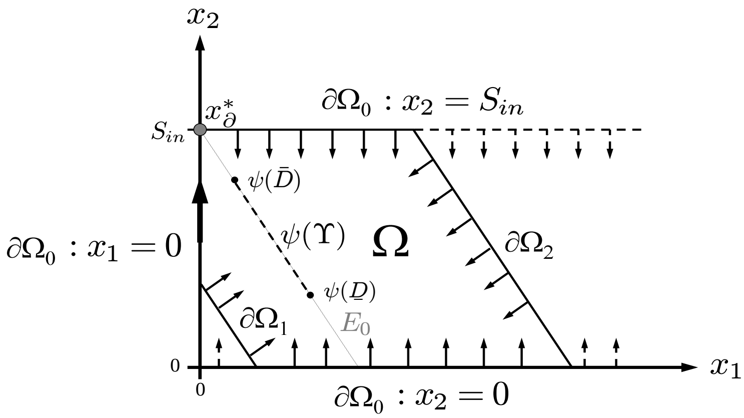

Figure 4. A graphical representation of Ω

whose characteristic polynomial is given by $P_\psi(\lambda; u) = \lambda^2 + \beta_1(u)\lambda + \beta_0(u)$, with

$$\beta_j(u) = u^{1-j}[S_{in} - \phi(u)]\mu'(\phi(u)) + ju$$

$j = 0, 1$. Let us note that $S_{in} - \phi(u) > 0$ —according to (27) (recall that $\phi(u) = S_-^*$)— and $u > 0$, $\forall u \in \Upsilon$. On the other hand, after several basic developments we get $\mu'(\phi(u)) = \frac{u^2}{\mu_m K_I}\left(\frac{K_S K_I}{\phi^2(u)} - 1\right)$, and from (26) (recalling that $\phi(u) = S_-^*$) one sees that $\mu'(\phi(u)) > 0, \forall u \in \Upsilon$. Therefore, $\beta_j(u) > 0$, $j = 0, 1$, $\forall u \in \Upsilon$, and consequently both roots of P_ψ have negative real part uniformly in u on Υ, *i.e.* $A_\psi(u)$ is Hurwitz for all $u \in \Upsilon$. Thus, Assumption A4 is satisfied.

Let $u = \alpha \in \Upsilon$, α being a constant value. Observe from Remark 5 that $F(x, \alpha) \neq 0_2$, $\forall x \in \mathrm{int}(\Omega) \setminus \{\psi(\alpha)\}$, and consequently $\psi(\alpha)$ is the unique equilibrium point in $\mathrm{int}(\Omega)$. Let us, on the other hand, define $A_\partial(\alpha) = \frac{\partial F}{\partial x}(x_\partial^*, \alpha)$. A direct substitution into (31) yields

$$A_\partial(\alpha) = \begin{pmatrix} \mu(S_{in}) - \alpha & 0 \\ \frac{\mu(S_{in})}{Y} & -\alpha \end{pmatrix}$$

From this expression, one sees that $\mu(S_{in}) - \alpha$ is an eigenvalue of $A_\partial(\alpha)$. Since $\mu(S_{in}) - \alpha > 0$ (recall (14)), x_∂^* is unstable and consequently Assumption A5a is satisfied. Let us further define

$$V(x_1, x_2) = \frac{1}{2}\omega_0^2(x_1, x_2) \tag{32a}$$

with

$$\omega_0(x_1, x_2) = S_{in} - \frac{x_1}{Y} - x_2 \tag{32b}$$

The derivative of V along the system trajectories is given by $\dot{V}(x_1, x_2) = -\alpha\omega_0^2(x_1, x_2)$. This shows that the manifold $E_0 = \{(x_1, x_2) \in \Omega : \omega_0(x_1, x_2) = 0\}$ is attractive, *i.e.* $x(t; x_0) \to E_0$ as $t \to \infty$, $\forall x_0 \in \Omega$. Then, invariant sets in Ω ought to be in E_0 —actually $\psi(\Upsilon) \subset E_0$— and since E_0 is 1-dimensional, limit cycles in Ω may not exist. Thus, Assumption A5b is satisfied. Let us further note that, according to La Salle's invariance principle, for any $x_0 \in \Omega$, the system trajectories converge to the largest invariant set in E_0. The consideration of the system dynamics on this manifold yields: $\dot{x}_2 = -(S_{in} - x_2)[\mu(x_2) - \alpha]$. Since $(x_2 - \phi(\alpha))(S_{in} - x_2)[\mu(x_2) - \alpha] > 0$, $\forall [0, S_{in}) \setminus \{\phi(\alpha)\}$, we have that for all $x_0 \in E_0 \setminus \{x_\partial^*\}$: $x_2(t; x_0) \to \phi(\alpha)$ as $t \to \infty$ (see for instance [1, Example 4.2]), or equivalently $x(t; x_0) \to \psi(\alpha)$ as $t \to \infty$, $\forall x_0 \in E_0 \setminus \{x_\partial^*\}$. This does not only corroborate the absence of limit cycles in Ω but also shows (under the consideration of La Salle's invariance principle and the unattractive nature of x_∂^*) that $x(t; x_0) \to \psi(\alpha)$ as $t \to \infty$, $\forall x_0 \in \mathrm{int}(\Omega)$.

From (15b), (29), and Remark 5, one sees that $\phi(u) \triangleq h(\psi(u))$ is indeed given by the expression in (30). Furthermore, from Remark 6, one corroborates that $\phi(u)$ is a monotonic —actually, strictly increasing— function on Υ. Thus, Assumption A6 is satisfied.

Since Assumptions A1–A6 are satisfied, the control design methodology of Proposition 1 may be suitably applied to the bioreactor model for the regulation of the substrate concentration (by means of the dilution rate D). Observe that since $\phi'(u) > 0$, $\forall u \in \Upsilon$, we have that $s = 1$ in the auxiliary dynamics, *i.e.* in (3).

Remark 7
Consider the closed-loop system (1)–(3) that takes place from the application of the proposed methodology to the bioreactor model, *i.e.* with f, g, and h as defined in Eqs. (15). Let us reconsider the scalar function in Eq. (32a). Its derivative along the closed-loop system trajectories is given by $\dot{V}(x_1, x_2, \theta) = -\theta\omega_0^2(x_1, x_2) \leq -\underline{D}\omega_0^2(x_1, x_2)$, $\forall(x_1, x_2, \theta) \in \Omega \times \Upsilon$, with $\omega_0(x_1, x_2)$ as defined in Eq. (32b). This shows that the manifold $E_c = \{(x_1, x_2, \theta) \in \Omega \times \Upsilon : \omega_0(x_1, x_2) = 0\}$ is attractive, *i.e.* $(x, \theta)(t) \to E_c$ as $t \to \infty$, $\forall(x, \theta)(0) \in \Omega \times \Upsilon$. Then, invariant sets in $\Omega \times \Upsilon$ ought to be in E_c; actually $\varphi(\mathcal{R}) \subset E_c$. The consideration of the closed-loop system dynamics on this manifold yields

$$\dot{x}_2 = -(S_{in} - x_2)[\mu(x_2) - \theta]$$
$$\dot{\theta} = k\eta(\theta)(y_d - x_2)$$

Let us define the scalar function

$$V_c(x_2, \theta) = \int_{y_d}^{x_2} \varpi_0(\zeta)d\zeta + \int_{D_d}^{\theta} \varpi_1(\vartheta)d\vartheta$$

where

$$\varpi_0(x_2) \triangleq \frac{x_2 - y_d}{S_{in} - x_2} \quad , \quad \varpi_1(\theta) \triangleq \frac{\theta - D_d}{k\eta(\theta)}$$

and $D_d \triangleq \mu(y_d)$. Observing that $(x_2 - y_d)\varpi_0(x_2) > 0$, $\forall x_2 \in [0, S_{in}) \setminus \{y_d\}$, and $(\theta - D_d)\varpi_1(\theta) > 0$, $\forall \theta \in \Upsilon \setminus \{D_d\}$, we conclude by previous arguments that $V_c(x_2, \theta)$ is positive definite on E_c with respect to (y_d, D_d), *i.e.* $V_c(x_2, \theta) \geq 0$, $\forall(x_2, \theta) \in E_c$, with $V_c(x_2, \theta) = 0 \iff (x_2, \theta) = (y_d, D_d)$. Its derivative along the closed-loop system dynamics on E_c is given by $\dot{V}_c(x_2, \theta) = -(x_2 - y_d)[\mu(x_2) - D_d]$ whence we see that $\dot{V}_c(x_2, \theta) \leq 0$, $\forall(x_2, \theta) \in E_c$, with $\dot{V}_c(x_2, \theta) = 0$ on $E_1 = \{(x_1, x_2, \theta) \in E_c : x_2 = y_d\} \supset \varphi(\mathcal{R})$. Hence, by La Salle's invariance principle, for any $(x_1, x_2, \theta) \in E_c$, the closed-loop system trajectories approach the largest invariant in E_1 asymptotically in time. Since E_1 is 1-dimensional, we conclude that there cannot be limit cycles in E_c, and there are consequently not in $\Omega \in \Upsilon$. Furthermore, from La Salle's invariance principle and the unstable nature of the invariant sets on $\partial(\Omega \times \Upsilon)$, we conclude that for any $(x, \theta)(0) \in \text{int}(\Omega \times \Upsilon)$, we have that $(x, \theta)(t) \to \varphi(y_d)$ as $t \to \infty$, with $(x, \theta)(t) \in \text{int}(\Omega \times \Upsilon)$, $\forall t \geq 0$. Since this does not depend on the value of k, the output feedback regulation is achieved, avoiding input saturation, for any $k > 0$.

D. DEVELOPMENTS OF SUBSECTION 4.2

Since the results in [10] are uniform in the system order n —*i.e.* they do not depend on the number of trays— let us consider the simplest case of Eqs. (17), which arises with $n = 3$, *i.e.*

$$H_1\dot{x}_1 = V\kappa(x_2) - Vx_1$$
$$H_2\dot{x}_2 = Lx_1 + V\kappa(x_3) - (L + F_f)x_2 - V\kappa(x_2) + F_f z_f$$
$$H_3\dot{x}_3 = (L + F_f)x_2 - (L + F_f - V)x_3 - V\kappa(x_3)$$

Under the state, input, and output variable definition previously stated (*i.e.* $x = (x_1, \ldots, x_n)$, $u = L$, and $y = x_1$), the system dynamics adopts the form of the state model in Eqs. (1), *i.e.* $\dot{x} = f(x) + g(x)u \triangleq F(x, u)$, with

$$f(x) = \begin{pmatrix} [V\kappa(x_2) - Vx_1]/H_1 \\ [V\kappa(x_3) - F_f x_2 - V\kappa(x_2) + F_f z_f]/H_2 \\ [F_f x_2 - (F_f - V)x_3 - V\kappa(x_3)]/H_3 \end{pmatrix}$$

$$g(x) = \begin{pmatrix} 0 \\ (x_1 - x_2)/H_2 \\ (x_2 - x_3)/H_3 \end{pmatrix} \quad \text{and} \quad h(x) = x_1$$

Further, from Fact F4 (see Appendix A), we have that

$$\phi'(u) = \frac{T_1(u) + T_2(u)}{D_J(u)}$$

with

$$T_1(u) = \big(u + F_f - V + V\kappa'(\psi_3(u))\big)\kappa'(\psi_2(u))\big(\psi_1(u) - \psi_2(u)\big)$$
$$T_2(u) = V\kappa'(\psi_2(u))\kappa'(\psi_3(u))\big(\psi_2(u) - \psi_3(u)\big)$$

and

$$D_J(u) = \big(u + F_f + (V-u)\kappa'(\psi_2(u))\big)(u + F_f - V) + (V-u)V\kappa'(\psi_2(u))\kappa'(\psi_3(u))$$

Finally, from these expressions, Remark 3, (20) (wherefrom we have that $V - u > 0$ and $u + F_f - V > 0$, $\forall u \in \Upsilon$), and (21), one sees that $\phi'(u) > 0$, $\forall u \in \Upsilon$.

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