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# Maximal unstable dissipative interval to preserve multi-scroll attractors via multi-saturated functions

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**Abstract**

In this paper, we present families of piecewise linear systems which are controlled by a continuous piecewise monoparametric control function for the generation of monoparametric families of multi-scroll attractors. Thus the maximum range of values that the parameter set can take in order to preserve the useful dynamics for generating of multi-scroll attractors is found and it will be called maximal robust dynamics interval (MDI). This class of dynamical systems is the results of combining two or more unstable "one-spiral" trajectories. We give necessary and sufficient conditions in order to preserve multiscroll attractors in terms of a parameter, i.e., a family of multi-scroll attractors is generated by means of a family of switching systems with multiple monoparametric companion matrices. Lastly, we provide an example to show how the developed theory works.

## 1 Introduction

Currently, generation of multi-scroll chaotic attractors has been extensively studied and it is no longer a very difficult task to find a set of fixed parameter values to generate multi-scroll attractors. Since the pioneering work of Suykens & Vandewalle [14, 15], several chaos generation mechanisms are further investigated by analyzing their trajectories and electronic implementation [13, 17]. For example, Lü *et al.* [9] introduced a hysteresis series switching approach for generating multi-scroll chaotic attractors, and the hysteresis series used is a discontinuous control function. Other systematic methods for generating multi-scroll chaotic attractors using discontinuous control function are dissipative linear systems with unstable dynamics

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(UDS's) approaches by Campos-Cantón *et al.* [6, 7, 12]. On the other hand, there are methods to create the multi-scroll chaotic attractors using continuous control functions, for example, by saturated functions [10] or threshold functions [11]. And so we find in the literature a lot of works dedicated to the generation of multi-scroll attractors [16, 18] based on piecewise linear systems or nonlinear systems by means of multiplying their states. In this work, we are interested in continuous piecewise functions (saturate functions) as controllers for the generation of monoparametric families of multi-scroll attractors.

There is a class of dynamical systems that have saddle equilibrium points as responsible for stretching and folding behavior displayed by multi-scroll attractors, i.e., chaotic behavior emerging from saddle equilibrium points in this kind of dynamical systems. Thereby, after previous works a question arises as follows: what is the maximum range of values that the parameter set can take to preserve the useful dynamics for the generation of multi-scroll attractors? This question was firstly answered by Aguirre-Hernández *et al.* [1] using discontinue control functions and UDS's in $\mathbb{R}^3$. The stable subspace of this kind of UDS's has stability index 1. Another form related with the stability index of UDS's is the instability index, in this case it is instability index 2. Discontinuous functions have been used to handle linear systems that have only saddle equilibria with the same stability index. So the MDI is only determined by this type of equilibria and families of multi-scroll attractor are defined based on segment of polynomials. On the other hand, continuous control functions can deal with saddle equilibria of UDS's with different stability index (1 and 2). So the multi-scroll attractors oscillate around these two type of saddle equilibria. In this work, we are interested in continuous piecewise functions (saturate functions) as controllers for the generation of monoparametric families of multi-scroll attractors.

In control theory applied to linear systems, families of polynomials have been widely studied with the purpose of stabilizing feedback controller systems, continuous or discrete time (see for instance [4, 5] and references therein). In the sense of stability of continuous-time linear systems, a polynomial is said to be *stable* if all of its roots have negative real part. These polynomials are also called *Hurwitz polynomials*, in reference of *Hurwitz stability* for continuous linear systems. On the other hand, a polynomial is said to be *unstable* if at least one root has positive real part. Similarly, in discrete-time linear systems, a polynomial is said to be *stable* if all of its roots have modulus less than one. These polynomials are also called *Schur polynomials* in reference to *Schur stability* for discrete linear systems. On the other hand, a polynomial is said to be *unstable* if at least one root has modulus greater than one. The target is the choice of one parameter required to hold stability of the control system, maintaining its spectrum in the stability zone, i.e., if the characteristic polynomial is stable then it is possible to perturb the system via a chosen parameter to find a ray or segment of stable polynomials as it is done in [2, 3]. The concept of rays and segments will be given in the section 2. In this way the maximal stability interval [4, 8] is determined by a family of stable characteristic polynomials. Thus, if the selected parameter value belongs to the maximal stability interval then the eigenvalues of the Jacobian matrix are contained in the stability zone. Our case study is when the parameter value takes values outside the maximal stability interval, so the dynamics of the system changes to be unstable which means that one or more eigenvalues are outside of the stability zone. Depending on the number of the eigenvalues inside and outside the stability zone is the class of dynamics obtained, for instance, dissipative systems with unstable dynamics (UDS's) have a saddle-focus equilibrium which is responsible for stable and unstable manifolds and the sum of its eigenvalues is negative. In $\mathbb{R}^3$, we consider two types of UDS's defined as follows: if the Jacobian of the system at the equilibrium point $x_0$ has eigenvalues $\lambda_i$, $i = 1, 2, 3$ with $\sum_{i=1}^{3} \lambda_i < 0$, then the system is said to be dissipative and it will be called UDS of *type I* (UDS-I) if the spectrum of the Jacobian of the system is comprised by one negative real eigenvalue and the other two are complex conjugate with positive real part; whilst it is said to be of *type II* (UDS-II) if one of its eigenvalues is positive real and the other two are complex conjugate with negative real part. Based on this class of dynamical systems is possible to generate a double-scroll attractors as the result of the combination of two unstable "one-spiral" trajectories.

In this work, a family of multi-scroll attractors generated by a family of switching systems with multiple monoparametric companion matrices is provided. A better description of the problem will be given in the section 2. The rest of the work is organized as follows: In section 2 we present the description and statement of the problem of generating multi-scroll attractors by switching UDS's. The section 3 contains a test to determine if a polynomial satisfies the conditions of instability and dissipativity. The section 4 contains the preliminaries of the dynamics behavior in closed-loop systems with a polynomial approach; the description of the class of controls and systems to be treated is given. In the section 5, we describe a technique to generate families of multiple-scrolls attractors via piecewise linear systems based on UDS. An illustrative example is given. Finally conclusions are drawn in Section 6.

## 2 Statement of the problem

The problem of generating chaotic attractors emerging from PWL systems has been addressed by constructing a monoparametric family of dynamical systems well called *control systems* of the form:

$$\dot{\chi} = \tilde{A}\chi + \tilde{b}u, \tag{1}$$

where $\chi \in \mathbb{R}^n$ is the state vector, $\tilde{A} \in \mathbb{R}^{n\times n}$ is a linear operator, $\tilde{b} \in \mathbb{R}^n$ is a constant vector. The characteristic polynomial of the matrix $\tilde{A}$ is $p_{\tilde{A}}(t) = t^n + a_1t^{n-1} + \cdots + a_{n-1}t + a_n$. It is well known that if the pair $(\tilde{A}, \tilde{b})$ is completely controllable, then there exists a change of coordinates $x = Q^{-1}\chi$ such that the system (1) is expressed as

$$\dot{\mathbf{x}} = A\mathbf{x} + bu, \tag{2}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \dots & -a_1 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

The most commonly used controllers for this class of systems are of the form $u = -kc^Tx$, with uncertain vector $c^T = (c_n, c_{n-1}, \dots, c_1)$. Then the characteristic polynomial of the closed loop system is given by $p_A(t) + kp_1(t)$, which is a ray of polynomials where $p_A(t)$ is the vertex and $p_1(t) = c_nt^{n-1} + c_{n-1}t^{n-2} + \cdots + c_1$ is the direction. If $p_A(t)$ is unstable then it is possible to perturb the system via the parameter $k$ to find a ray or a segment of the ray, consisting on unstable polynomials. That is, the maximal interval $(k^-_{\min}, k^+_{\max})$ for which the family $p_A(t) + kp_1(t)$ keeps the dynamics proportioned by $p_A(t)$ will be determined.

Our case study is when the $k$ parameter belongs to the maximal unstable interval, in order to generate a family of UDS's with saddle-focus equilibrium which is responsible for stable and unstable manifolds and the sum of its eigenvalues is negative. In order to generate a family of attractors via UDS's with multiple saddle hyperbolic equilibria $p_1, p_2, \dots, p_m$, we provide a family of piecewise affine continuous systems that generate multi-scroll attractors emerging from a system (2), endowed of the family of switching laws $u_j$

which arises piecewise linear systems of the form

$$\dot{\mathbf{x}} = \begin{cases} A_1(k)\mathbf{x} + bu_1, & \text{for } \mathbf{x} \in \mathcal{D}_{p_1}; \\ A_2(k)\mathbf{x} + bu_2, & \text{for } \mathbf{x} \in \mathcal{D}_{p_2}; \\ \vdots & \\ A_m(k)\mathbf{x} + bu_m, & \text{for } \mathbf{x} \in \mathcal{D}_{p_m}; \end{cases} \tag{3}$$

where $k \in (\underline{k}, \overline{k})$, $\mathcal{D}_{p_i} \subset \mathbb{R}^3$ are the domains for each $u_i$, $1, 2, \ldots, m$, such that $\cap_{i=1}^{m} \mathcal{D}_{p_i} = \emptyset$ and $\cup_{i=1}^{m} \mathcal{D}_{p_i} = \mathbb{R}^3$. The interval $(\underline{k}, \overline{k})$ is the maximal interval of dissipativity and robust dynamics, that is, the maximal interval of perturbation of the matrix $A$ for still having scroll attractors around the equilibria $p_i = -A^{-1}(k)bu_i$. To achieve the aforementioned aim we assume that the system (2) satisfies:

**DS1** is UDS at the equilibrium point $p = -A^{-1}bu \in \mathbb{R}^3$.

**DS2** is in controllable canonical form.

In the following sections we shall design the switching systems $A_i(k)x + bu_i$, $i = 1, \ldots, m$, for $m \geq 3$.

## 3 Hyperbolic equilibria of type I and II

The analysis begins with single tests to know if a polynomial satisfies the instability and dissipative conditions. To establish this characterization of the hyperbolic equilibria of type I and II it is necessary the following definition and result.

*Definition* 3.1. A 3-degree polynomial $p(t)$ will be called *dissipative* if it has roots $\lambda_i$, $i = 1, 2, 3$ with $\sum_{i=1}^{3} \lambda_i < 0$. A dissipative polynomial $p(t)$ will be called *UDS-I polynomial* if it has one negative real root and the other two are complex conjugate with positive real part; whilst it will be called *UDS-II polynomial* if one of its roots is positive real and the other two are complex conjugate with negative real part.

LEMMA (3.2). *The polynomial* $p(t) = t^3 + a_1t^2 + a_2t + a_3$ *has two pure imaginary roots if and only if* $a_2a_1 - a_3 = 0$ *and* $a_2 > 0$.

*Proof.* $(\Rightarrow)$ If $i\omega$ is a root of $p(t)$ then $a_3 - a_1\omega^2 + i\omega(a_2 - \omega^2) = 0$. It implies that $a_3 - a_1\omega^2 = 0$ and $a_2 = \omega^2$. Thus $a_2a_1 - a_3 = 0$ and $a_2 > 0$.

($\Leftarrow$) If $a_2a_1 - a_3 = 0$ and $a_2 > 0$ then

$$\begin{aligned} p(t) &= t^3 + a_1t^2 + a_2t + a_2a_1 \\ &= t(t^2 + a_2) + a_1(t^2 + a_2) \\ &= (t + a_1)(t^2 + a_2) \\ &= (t + a_1)(t + i\sqrt{a_2})(t - i\sqrt{a_2}) \end{aligned}$$

□

*Example* 3.3. Consider the polynomial $p(t) = t^3 - t^2 + 4t - 4$ we can see that $a_2 = 4 > 0$ and $a_2a_1 - a_3 = (4)(-1) - (-4) = 0$. Therefore, $p(t)$ has two pure imaginary roots.

The lemma 3.2 gives the possibility to determine the limits of the maximal unstable interval. We are interested in finding polynomials that guarantee the generation of UDS-I. So note that if $p(t) = t^3 + a_1t^2 + a_2t + a_3$ is a polynomial with $a_3 > 0$ and has two roots of the form $\alpha \pm i\beta$, $\alpha > 0$, from the analysis of the derivative of $p(t)$ and its discriminant $4a_1^2 - 12a_2$, $(a_1^2 - 3a_2)$, the following theorem is held.

LEMMA (3.4). *The polynomial $p(t) = t^3 + a_1t^2 + a_2t + a_3$ has a negative real root and two complex conjugate roots $\alpha \pm i\beta$ with $\alpha > 0$ and $\beta \neq 0$ if and only if $a_3 > 0$ and one of the following cases is satisfied:*

*a)* $a_1^2 - 3a_2 \leq 0$ *and* $[a_2a_1 \leq 0$ *or* $a_2a_1 - a_3 < 0]$;

*b)* $a_1^2 - 3a_2 > 0$, $\frac{-a_1 + \sqrt{a_1^2 - 3a_2}}{3} < 0$ *and*

$$p\left(\frac{-a_1 - \sqrt{a_1^2 - 3a_2}}{3}\right) > 0;$$

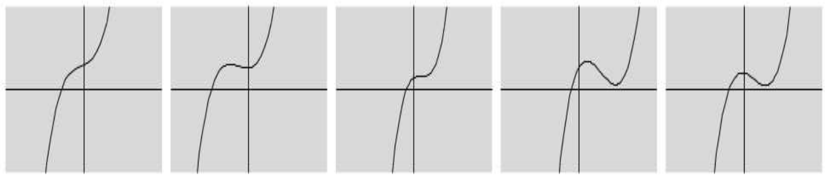
*c)* $a_1^2 - 3a_2 > 0$, $\frac{-a_1 - \sqrt{a_1^2 - 3a_2}}{3} > 0$ *and*

$$p\left(\frac{-a_1 + \sqrt{a_1^2 - 3a_2}}{3}\right) > 0;$$

*d)* $a_1^2 - 3a_2 > 0$, $\frac{-a_1 - \sqrt{a_1^2 - 3a_2}}{3} < 0$, $\frac{-a_1 + \sqrt{a_1^2 - 3a_2}}{3} > 0$ *and*

$$p\left(\frac{-a_1 \pm \sqrt{a_1^2 - 3a_2}}{3}\right) > 0.$$

*Proof.* ⇒]. If $p(t) = t^3 + a_1t^2 + a_2t + a_3$ $(a_3 > 0)$ has a negative real root and two complex conjugate roots $\alpha \pm i\beta$ with $\alpha > 0$ and $\beta \neq 0$ then the graph of $p(t)$ can have the following forms (see Figure 1):
The graphs are determined from the behavior of the derivative $p'(t)$: In the



(a) graph 1 (b) graph 2 (c) graph 3 (d) graph 4 (e) graph 5

Figure 1: Graph of $p(t)$.

graphs 1 and 3 the function $p(t)$ is increasing and then $p'(t) = 3t^2+2a_1t+a_2$ is positive for every value of $t$ whence $a_1^2 - 3a_2 \leq 0$; consequently a) is obtained.

The graph 2 indicates that $p(t)$ has two negative critical points (when $p'(t) = 0$) that determines both local minimum and maximum values where the local maximum value is negative, that is, $a_1^2 - 3a_2 > 0$, $\frac{-a_1+\sqrt{a_1^2-3a_2}}{3} < 0$ and $p(\frac{-a_1-\sqrt{a_1^2-3a_2}}{3}) > 0$; thence b) is obtained.

The graph 4 indicates that $p(t)$ has two positive critical points (when $p'(t) = 0$) that determines a local minimum value and a local maximum value which the local minimum value is positive, that is, $a_1^2 - 3a_2 > 0$, $\frac{-a_1-\sqrt{a_1^2-3a_2}}{3} > 0$ and $p(\frac{-a_1+\sqrt{a_1^2-3a_2}}{3}) > 0$ and then c) is obtained.

The graph 5 indicates that $p(t)$ has a negative critical point and one positive (when $p'(t) = 0$) that determines a local maximum value and a local minimum value which both are positive, that is, $a_1^2-3a_2 > 0$, $\frac{-a_1-\sqrt{a_1^2-3a_2}}{3} < 0$, $\frac{-a_1+\sqrt{a_1^2-3a_2}}{3} > 0$ and $p(\frac{-a_1\pm\sqrt{a_1^2-3a_2}}{3}) > 0$ and then d) is obtained.
⇐] Suppose that a) is satisfied, then $p'(t) \geq 0$ for every $t \in \mathbb{R}$, then $p(t)$ is an increasing function. Since $a_3 > 0$ we obtain the graph 1 and consequently $p(t)$ has a negative root and two roots $\alpha + i\beta$ with $\alpha > 0$ and $\beta \neq 0$. Similar proofs can be obtained for other cases. □

*Example* 3.5. If we consider the following polynomial $p_0(t) = t^3+t^2+4t+30$, we can see that it satisfies

1. $a_3 = 30 > 0$,
2. $4a_1^2 - 12a_2 = -52 < 0$,
3. $a_2 a_1 - a_3 = -26 < 0$.

That is, $p_0(t)$ satisfies condition a) from lemma 3.4, so it has one real negative root and two roots in the form $\alpha + i\beta$ with $\alpha > 0$ and $\beta \neq 0$.

Similarly, by taking $f(t) = p(-t)$ the following result, in order to characterize the hyperbolic points of type II is held.

LEMMA (3.6). *The polynomial $f(t) = t^3 + b_2 t^2 + b_1 t + b_0$ has a positive real root and two complex roots in the form $\alpha \pm i\beta$, $\alpha < 0$ if and only if one of the following is satisfied:*

*a)* $b_0 < 0$, $b_2^2 - 3b_1 \leq 0$ *and* $[b_1 b_2 \geq 0$ *or* $b_0 - b_1 b_2 < 0]$;

*b)* $b_0 < 0$, $b_2^2 - 3b_1 > 0$, $\frac{-b_2+\sqrt{b_2^2-3b_1}}{3} < 0$ *and*

$$f\left(\frac{-b_2 - \sqrt{b_2^2 - 3b_1}}{3}\right) < 0;$$

*c)* $b_0 < 0$, $b_2^2 - 3b_1 > 0$, $\frac{-b_2-\sqrt{b_2^2-3b_1}}{3} > 0$ *and*

$$f\left(\frac{-b_2 + \sqrt{b_2^2 - 3b_1}}{3}\right) < 0;$$

*d)* $b_0 < 0$, $b_2^2 - 3b_1 > 0$, $\frac{-b_2-\sqrt{b_2^2-3b_1}}{3} < 0$, $\frac{-b_2+\sqrt{b_2^2-3b_1}}{3} > 0$ *and*

$$f\left(\frac{-b_2 \pm \sqrt{b_2^2 - 3b_1}}{3}\right) < 0.$$

*Proof.* It is similar to the proof of lemma 3.4. □

## 4 The maximal UDS interval

As it has been pointed out, the object of study in this work is the generation of strange attractors from point of view of the control design theory by considering the control system

$$\dot{\mathbf{x}} = A\mathbf{x} + bu, \tag{4}$$

where $b \in \mathbb{R}^n$ and the pair $(A, b)$ is in controllable canonical form. Besides, if the linear control $u(k, \mathbf{x}) = -kc^T\mathbf{x}$ is considered, with $c^T = (c_n, c_{n-1}, \ldots, c_1)$, then the characteristic polynomial of the closed loop system is $P(t, k) = p_0(t) + kp_1(t)$, where $k$ is a real parameter, $p_0(t)$ is the characteristic polynomial of $A$ and $p_1(t) = c_n t^{n-1} + c_{n-1}t^{n-2} + \cdots + c_1$ is an arbitrary polynomial. If we consider $p_0(t)$ as a stable polynomial (its $n$ roots contained in the open left half plane) then the polynomial family $\mathbb{P}(t, 0) = p_0(t)$ is stable. Thus, we can perturb the variable $k$ around zero for keeping the stability of the system. Such a problem is called the problem of finding the maximal stability interval and was studied by Bialas around 1985 [4] and recently by López-Renteria *et al* in [8]. Related problems were studied in [2, 3].

The aim is to establish a similar result for the family of polynomials $P(t, k) = p_0(t) + kp_1(t)$ for which $p_0(t)$ has $n_1$ roots in $\mathbb{C}^-$ and $n - n_1$ roots in $\mathbb{C}^+$ for all $k$ in the maximal perturbed interval around zero. This is the problem of finding the maximal robust dynamics interval with a polynomial approach and it was solved in [1].

The following results concerning to the maximal UDS interval are reported in the above cited reference.

*Definition* 4.1. *A robust dynamics interval* of the monoparametric family $p_k(t)$ is an interval $[a, b]$ if $p_k(t)$ has $n_1$ roots in $\mathbb{C}^-$ and $n - n_1$ roots in $\mathbb{C}^+$ for all $k \in [a, b]$. The greatest of these intervals will be called the *maximal robust dynamics interval.*

Consider $p_0(-i\omega) = P(\omega^2) - i\omega Q(\omega^2)$ and $p_1(i\omega) = p(\omega^2) + i\omega q(\omega^2)$, then

$$P(i\omega, k)p_0(-i\omega) = G(\omega) + kF(\omega) + ik\omega H(\omega),$$

where

$$\begin{aligned}
F(\omega) &= p(\omega^2)P(\omega^2) + \omega^2 q(\omega^2)Q(\omega^2),\\
G(\omega) &= P^2(\omega^2) + \omega^2 Q^2(\omega^2),\\
H(\omega) &= q(\omega^2)P(\omega^2) - p(\omega^2)Q(\omega^2).
\end{aligned}$$

Now define, for an arbitrary polynomial $f(t)$, the set

$$R(f) = \{\xi \in \mathbb{C} : f(\xi) = 0\}.$$

Let $R(f)_{\mathbb{R}^+}$ denotes the set of positive real elements of $R(f)$ and now we define the sets

$$\begin{aligned} K^+ &= \{F(\omega_l) \,:\, \omega_l \in R(H)_{\mathbb{R}^+} \cup \{0\},\ F(\omega_l) > 0\}, \\ K^- &= \{F(\omega_l) \,:\, \omega_l \in R(H)_{\mathbb{R}^+} \cup \{0\},\ F(\omega_l) < 0\}. \end{aligned}$$

If there are no elements in $R(H)_{\mathbb{R}^+} \cup \{0\}$ such that $F(\omega_l) > 0$, then we define $K^+ = \{0^+\}$ . Similarly, if $R(H)_{\mathbb{R}^+} \cup \{0\}$ does not contain elements such that $F(\omega_l) < 0$, so we define $K^- = \{0^-\}$. With the aforementioned the following result for a polynomial family is held.

Theorem (4.2). *(Maximal robust dynamics interval) Consider the polynomial family $P(t,k) = p_0(t) + kp_1(t)$, where $p_0(t)$ is a $n-$degree polynomial with $n_1$ roots in $\mathbb{C}^-$ and $n - n_1$ roots in $\mathbb{C}^+$. Suppose the $n > deg\, p_1(t)$ and let $F(\omega)$, $G(\omega)$ and $H(\omega)$ be the polynomials defined above. Then $P(t,k)$ has $n_1$ roots in $\mathbb{C}^-$ and $n - n_1$ roots in $\mathbb{C}^+$ for all $k \in (k^-_{\min}, k^+_{\max})$, where*

$$\begin{aligned} k^-_{\min} &= \max\left\{-\frac{G(\omega_l)}{F(\omega_l)} \,:\, F(\omega_l) \in K^+\right\}, \\ k^+_{\max} &= \min\left\{-\frac{G(\omega_l)}{F(\omega_l)} \,:\, F(\omega_l) \in K^-\right\}. \end{aligned}$$

If $K^+ = \{0^+\}$ ($K^- = \{0^-\}$, resp.) then $k^-_{\min} = -\infty$ ($k^+_{\max} = \infty$, resp.).

In order to create a family of USD systems, the maximal interval of "saddleness" has been obtained and we just need the condition of negativity for the roots sum which is the dissipativity condition (also given in [1]).

Lemma (4.3). *The sum of the roots of the polynomial $p(t) = a_0t^n + a_1t^{n-1} + \cdots + a_{n-1}t + a_n$ is negative if and only if $\frac{a_1}{a_0} > 0$.*

The above results are sufficient to establish the maximal UDS interval.

Theorem (4.4). *Consider the control system*

$$\dot{\mathbf{x}} = A\mathbf{x} + bu,$$

*where $A \in \mathbb{R}^{n\times n}$, $b, \mathbf{x} \in \mathbb{R}^n$ and $u = -kc^T\mathbf{x}$, with $k \in \mathbb{R}$ and $c^T = (c_n, c_{n-1}, \ldots, c_1)$. If the characteristic polynomial of $A$, $p_0(t) = t^n + a_1t^{n-1} +$*

$\cdots + a_n$, *has* $n_1$ *roots in* $\mathbb{C}^-$ *and* $n - n_1$ *roots in* $\mathbb{C}^+$, *then the maximal robust dynamics interval is given by* $K = S \cap (k^-_{\min}, k^+_{\max})$, *where* $S = \{k \in \mathbb{R} : a_1 + kc_1 > 0\}$.

# 5 A family of multi-scroll attractors

In this section a family of multiple scrolls attractors will be generated based on piecewise linear system which will be designed with changing companion matrix depending on a real parameter $k$ by using a multi-saturating control.

Let us to consider the control system

$$\dot{\mathbf{x}} = A\mathbf{x} + bu, \tag{5}$$

with $A \in \mathbb{R}^{3\times3}$ and $\mathbf{x}, b \in \mathbb{R}^3$ and $u \in \mathbb{R}$ is a saturated control given by $u = -kc^T\mathbf{x} - f_s(c^T\mathbf{x})$, where $k \in \mathbb{R}$, $c^T = (c_3, c_2, c_1)$ and $f_s(c^T\mathbf{x})$ is the saturated function

$$f_s(c^T\mathbf{x}) = \begin{cases} w, & \text{for} \quad v < c^T\mathbf{x}; \\ \mu c^T\mathbf{x}, & \text{for} \quad |c^T\mathbf{x}| \leq v; \\ -w, & \text{for} \quad c^T\mathbf{x} < -v. \end{cases} \tag{6}$$

Suppose that the system satisfies the hypotheses **DS1** and **DS2**. Hypothesis **DS2** allows to consider the system (5) in controllable canonical form, that is,

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}, \text{ and } b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

where its characteristic polynomial is $P_A(t) = t^3 + a_1t^2 + a_2t + a_3$. It is not hard to see that

$$A^{-1} = \begin{bmatrix} -\frac{a_2}{a_3} & -\frac{a_1}{a_3} & -\frac{1}{a_3} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

and if $\mathbf{x} = (x_1, x_2, x_3)^T$, the equilibrium $\mathbf{x}^* = -A^{-1}bu = (-\frac{u}{a_3}, 0, 0)^T$ is located just in the $x_1$-axis. Thus, the plane $c^T\mathbf{x} = v$ can be relaxed and just consider the planes of commutation $c_3x_1 = v$ and $c_3x_1 = -v$ (with $c_1 = c_2 = 0$) which they are orthogonal to $x_1$-axis. Therefore, the saturated control is $u = -kc_3x_1 - f_s(x_1)$ and $f_s$ is given by the following saturated function:

$$f_s(x_1, w) = \begin{cases} w, & \text{for} \quad v/c_3 < x_1; \\ \mu c_3 x_1, & \text{for} \quad |x_1| \leq v/c_3; \\ -w, & \text{for} \quad x_1 < -v/c_3; \end{cases} \tag{7}$$

where $\mu$ is the slope of the middle segment, the upper radial $\{f_s(x_1, w) = w : x_1 \geq v/c_3\}$ and the lower radial $\{f_s(x_1, w) = -w : x_1 \leq -v/c_3\}$ are called *saturated plateaus*, and the segment $\{f_s(x_1, \mu) = \mu x_1 : |x_1| \leq v/c_3\}$ between two saturated plateaus is called the *saturated slope*. For $k = 0$ the system is able to generate chaotic attractor via saturated function and the system commutes between UDS-I and UDS-II for the saturated plateaus and saturated slope, respectively (see [10], for instance). The closed loop system (5) with the scalar feedback control $u(\mathbf{x}, k) = -kc^T\mathbf{x} - f_s(x_1, w)$ is given by

$$\dot{\mathbf{x}} = (A - kbc^T)\mathbf{x} - f_s(x_1, w)b. \tag{8}$$

The characteristic polynomial of the saturate plateaus $A + kbc^T$ as given above is $P_{SP}(t, k) = t^3 + t^2 + t + (a_3 + kc_3)$ and corresponds to UDS type I. In a similar way the characteristic polynomial of the saturate slope is $P_{SS}(t, k) = t^3 + a_1t^2 + a_2t + (a_3 + kc_3 + \mu c_3)$. The part of saturated slope can modify the stability of the system and corresponds to UDS type II. Thus the feedback control can modify the stability of the system and destroys the attractor, then the previous results about the maximal unstability interval need to be employed to warranty multiscroll behavior.

In order to achieve the emergence of multiple equilibria where the system behaves chaotically, a multi-saturated function series is considered instead of saturated function. Firstly, let us to consider two sets, the former is the set of saturation values $\Lambda_w = \{w_1, w_2, \ldots, w_d\}$ and the second is the set of commutation values $\Lambda_v = \{v_1, v_2, \ldots, v_r\}$ such that $v_1 < v_2 < \ldots < v_r$, and $w_1 < w_2 < \cdots < w_d$, with $d = 2(r - 1)$ and $r, d \in \mathbb{N}$. The cardinality of the set $\Lambda_w$ is the number of scrolls in the attractor. Thus with these two sets, it is now possible to define the piecewise continuous multi-saturated function as follows:

$$f_s(x_1, \Delta_w, \Delta_v) = \begin{cases} w_d, & \text{for } v_r/c_3 \leq x_1; \\ \mu_{d-1}c_3x_1 + \beta_{d-1}, & \text{for } v_{r-1}/c_3 \leq x_1 < v_r/c_3; \\ \vdots & \\ \mu_2c_3x_1 + \beta_2, & \text{for } v_3/c_3 \leq x_1 < v_4/c_3; \\ w_2, & \text{for } v_2/c_3 \leq x_1 < v_3/c_3; \\ \mu_1c_3x_1 + \beta_1, & \text{for } v_1/c_3 \leq x_1 < v_2/c_3; \\ w_1, & \text{for } x_1 < v_1/c_3; \end{cases} \tag{9}$$

with

$$\mu_1 = \frac{w_2 - w_1}{v_2 - v_1},\ \mu_2 = \frac{w_3 - w_2}{v_4 - v_3},\ \dots\ , \mu_{d-1} = \frac{w_d - w_{d-1}}{v_r - v_{r-1}},$$

and $\beta_1 = -\mu_1 v_1 + w_1, \beta_2 = -\mu_2 v_3 + w_2, \dots, \beta_{d-1} = -\mu_{d-1}v_{r-1} + w_{d-1}$.

Next, the system (5) with the multi-saturated control $u = -kc^T\mathbf{x} - f_s(x_1, \Delta_w, \Delta_v)$ is given as follows:

$$\dot{\mathbf{x}} = \begin{cases} (A - kbc^T)\mathbf{x} - w_d b, & \text{for} \quad v_r/c_3 \leq x_1; \\ (A - (k + \mu_{d-1})bc^T)\mathbf{x} - \beta_{d-1}b, & \text{for} \quad v_{r-1}/c_3 \leq x_1 < v_r/c_3; \\ \vdots & \\ (A - (k + \mu_2)bc^T)\mathbf{x} - \beta_2 b, & \text{for} \quad v_3/c_3 \leq x_1 < v_4/c_3; \\ (A - kbc^T)\mathbf{x} - w_2 b, & \text{for} \quad v_2/c_3 \leq x_1 < v_3/c_3; \\ (A - (k + \mu_1)bc^T)\mathbf{x} - \beta_1 b, & \text{for} \quad v_1/c_3 \leq x_1 < v_2/c_3; \\ (A - kbc^T)\mathbf{x} - w_1 b, & \text{for} \quad x_1 < v_1/c_3. \end{cases} \tag{10}$$

Then, there are two classes of equilibria which are described by the following sets:

$$\begin{aligned} A_{x_1} &= \left\{ w_j \left(A - kbc^T\right)^{-1} b = \left( \frac{w_j}{a_3 + kc_3}, 0, 0 \right)^T \right\}_{j=1}^{d}, \\ B_{x_1} &= \left\{ \beta_j \left(A - (k + \mu_j)bc^T\right)^{-1} b = \left( \frac{\beta_j}{a_3 + (k + \mu_j)c_3}, 0, 0 \right)^T \right\}_{j=1}^{d-1}. \end{aligned}$$

The characteristic polynomial of saturate plateaus $A - kbc^T$ is the same $P_{SP}(t,k) = t^3 + a_1t^2 + a_2t + (a_3 + kc_3)$, whilst the characteristic polynomial of saturate slopes $A - kbc^T + \mu_j bc^T$ are $P_{SS}(t,k) = t^3 + a_1t^2 + a_2t + (a_3 + kc_3 + \mu_j c_3)$, $j = 1, \dots, d-1$. Recall that theorem 4.2 shows how to find the maximal robust dynamics interval for $P_{SP}$ and lemma 4.3

gives the dissipativity interval. Consequently, theorem 4.4 allows us to keep unstability-dissipativity in a maximal interval, no matter what type of UDS in the open-loop system. Hereafter, we suppose that the equilibria set $A_{x_1}$ are of type I. However, it is necessary to guarantee the change of unstability of saturate slopes polynomial characteristic from type I to II.

Lemma (5.1). *Consider the polynomial $P_{SS}(t,k) = t^3 + a_1t^2 + a_2t + (a_3 + kc_3 + \mu_j c_3)$ with $a_1 > 0$. Suppose that for $k = -\mu$, $P_{SS}$ has one negative root and a pair of complex roots with positive real part (UDS I). Then $P_{SS}$ has one positive root and two complex roots with negative real part (UDS II) if and only if $a_1^2 - 3a_2 > 0$ and $k \in K_B^j$, where*

$$K_B^j = \begin{cases} k < k_{1,j}, & \text{if} \quad D < 0; \\ k < k_{2,j}, & \text{if} \quad \overline{D} > 0; \\ k < \min\{k_1, k_2\}, & \text{if} \quad \overline{D} < 0 \text{ and } D > 0; \end{cases} \tag{11}$$

*with*

$$\begin{aligned} k_{1,j} &= \min\left\{-\frac{1}{c_3}(D^3 + a_1D^2 + a_2D + a_3 + \mu_j c_3), -\frac{a_3}{c_3} - \mu_j\right\}, \\ k_{2,j} &= \min\left\{-\frac{1}{c_3}(\overline{D}^3 + a_1\overline{D}^2 + a_2\overline{D} + a_3 + \mu_j c_3), -\frac{a_3}{c_3} - \mu_j\right\}, \end{aligned}$$

*and*

$$\begin{aligned} D &= \frac{-a_1 + \sqrt{a_1^2 - 3a_2}}{3}, \\ \overline{D} &= \frac{-a_1 - \sqrt{a_1^2 - 3a_2}}{3}. \end{aligned}$$

*Proof.* The proof shall be based on lemma 3.6. Since $P_{SS}(t, -\mu)$ satisfies one of the hypothesis of lemma 3.4, then it can not satisfy item a) in lemma 3.6 for all $k$. However, condition $a_1^2 - 3a_2 > 0$ is the same in both lemmas. Now, condition $a_3 + kc_3 + \mu_j c_3 < 0$ is satisfied if and only if $k < -\frac{a_3}{c_3} - \mu_j$. Next, if $D = \frac{-a_1+\sqrt{a_1^2-3a_2}}{3}$ and $\overline{D} = \frac{-a_1-\sqrt{a_1^2-3a_2}}{3}$, we see that $P(D,k) < 0$ if and only if $k < -\frac{1}{c_3}(D^3 + a_1D^2 + a_2D + a_3 + \mu_j c_3)$. Similarly, $P(\overline{D},k) < 0$ is satisfied if and only if $k < -\frac{1}{c_3}(\overline{D}^3 + a_1\overline{D}^2 + a_2\overline{D} + a_3 + \mu_j c_3)$. Define

$$\begin{aligned} k_{1,j} &= \min\left\{-\frac{1}{c_3}(D^3 + a_1D^2 + a_2D + a_3 + \mu_j c_3), -\frac{a_3}{c_3} - \mu_j\right\}, \\ k_{2,j} &= \min\left\{-\frac{1}{c_3}(\overline{D}^3 + a_1\overline{D}^2 + a_2\overline{D} + a_3 + \mu_j c_3), -\frac{a_3}{c_3} - \mu_j\right\}. \end{aligned}$$

Thence, item b) in lemma 3.6 is satisfied if and only if $D<0$ and $k<k_1$; item c) is satisfied if $\overline{D}>0$ and $k<k_2$, and item d) is verifiable if and only if $k<\min\{k_1,k_2\}$, $\overline{D}<0$ and $D>0$. Finally, the change of unstability is given by

$$K_B^j=\begin{cases} k<k_{1,j}, & \text{if} \quad D<0;\\ k<k_{2,j}, & \text{if} \quad \overline{D}>0;\\ k<\min\{k_1,k_2\}, & \text{if} \quad \overline{D}<0 \text{ and } D>0; \end{cases} \tag{12}$$

as we claim. □

Let $K_{A_{x_1}}$ be the maximal UDS I interval for saturated plateaus. Then all the slopes of saturation are UDS II if $k\in K_{B_{x_1}}=\bigcap_{j=1}^{d-1}K_B^j$. With all the above discussion, the following result is held.

Theorem (5.2). *Consider the 3D control system*

$$\dot{x}=Ax+bu \tag{13}$$

*satisfying the hypothesis* **DS1** *and* **DS2**. *Let* $u=-kc_3x_1-f_s(x_1,\Delta_w,\Delta_v)$ *be a saturated control with* $f_s(x_1,\Delta_w,\Delta_v)$ *the multi-saturating function* (9). *Then the closed-loop system* (1) *possesses equilibria sets* $A_{x_1}$ *of type UDS I and* $B_{x_1}$ *of type UDS II if and only if* $k\in K_{\max}=K_{A_{x_1}}\cap K_{B_{x_1}}$. *Moreover, it is possible to generate a* $k-$*family of* $d-$*scroll attractors emerging from the equilibria set* $A_{x_1}$.

With the interest of showing how the proposed theory works we have taking the numerical example given in [10], where the proposed approach may generate 1-D $n$-scrolls, 2-D $n\times m$-grid scrolls, and 3-D $n\times m\times l$-grid scrolls chaotic attractors. Thus the example is given as follows.

*Example* 5.3. Consider the system

$$\begin{aligned}\dot{x} &= A_Ix+bu \\ &= \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -0.7 & -0.7 & -0.7\end{bmatrix}x+\begin{bmatrix}0\\0\\1\end{bmatrix}u.\end{aligned} \tag{14}$$

The characteristic polynomial of the open-loop system is $p_{A_I}(t)=t^3+0.7t^2+0.7t+0.7$ and by $a$) in lemma 3.4 we get that

$$\begin{aligned} a_3 &= 0.7>0,\\ 4a_1^2-12a_2 &= -6.44\le 0,\\ a_1a_2-a_3 &= -0.21<0.\end{aligned}$$

It implies that the system has a negative real eigenvalue and a pair of complex conjugate eigenvalues in the form $\alpha \pm i\beta$, with $\alpha > 0$. Next, by Lemma 4.3 the sum of its eigenvalues is negative due to $a_1 = 0.7 > 0$. Therefore, the system (14) is UDS-I.
The saturation function implemented [10] is given as follows:

$$f_s(x_1) = \begin{cases} 7, & \text{if } x_1 > 1; \\ 7x_1, & \text{if } |x_1| \leq 1; \\ -7, & \text{if } x_1 < -1. \end{cases} \tag{15}$$

From this saturated function is possible to generate the multi-saturated function as follows:

$$f_s(x_1) = \begin{cases} 35, & \text{if } 41 \leq x_1; \\ 7x_1 - 252, & \text{if } 39 \leq x_1 < 41; \\ 21, & \text{if } 21 \leq x_1 < 39; \\ 7x_1 - 126, & \text{if } 19 \leq x_1 < 21; \\ 7, & \text{if } 1 \leq x_1 < 19; \\ 7x_1, & \text{if } -1 \leq x_1 < 1; \\ -7, & \text{if } x_1 < -1. \end{cases} \tag{16}$$

The system (14) is UDS-I in the saturated plateaus, however the system (14) changes to UDS-II as it was mentioned previously due to saturated slope to the following matrix $A_{II}$:

$$A_{II} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 6.3 & -0.7 & -0.7 \end{pmatrix} \tag{17}$$

Due to the equilibria are situated on the $x_1-$axis we may take $c_1 = c_2 = 0$ and $c_3 = 1$; also with the sets $\Lambda_v = \{-1, 1, 19, 21, 39, 41\}$ and $\Lambda_w = \{-7, 7, 21, 35\}$ we get $\mu_0 = 7$ and the another part of the feedback $u =$

$-kx_1 - f_s(x_1)$ leaves the system (14) as follows

$$\dot{\mathbf{x}} = \begin{cases} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ (-0.7-k) & -0.7 & -0.7 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 35 \end{bmatrix}, & \text{if } 41 \leq x_1; \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ (6.3-k) & -0.7 & -0.7 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ -252 \end{bmatrix}, & \text{if } 39 \leq x_1 < 41; \\ \vdots & \vdots \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ (-0.7-k) & -0.7 & -0.7 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix}, & \text{if } 1 \leq x_1 < 19; \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ (6.3-k) & -0.7 & -0.7 \end{bmatrix} \mathbf{x}, & \text{if } -1 \leq x_1 < 1; \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ (-0.7-k) & -0.7 & -0.7 \end{bmatrix} \mathbf{x} - \begin{bmatrix} 0 \\ 0 \\ -7 \end{bmatrix}, & \text{if } x_1 < -1. \end{cases} \tag{18}$$

The following is the computing of the maximal UDS-I interval: For the polynomials $p_{A_I}(t) = t^3 + 0.7t^2 + 0.7t + 0.7$ and the polynomial $p_1(t) = t^2$ we have that

$$\begin{aligned} p_{A_I}(i\omega) &= (0.7 - 0.7\omega^2) + i\omega(0.7 - \omega^2), \\ p_1(i\omega) &= -\omega^2, \end{aligned}$$

for which

$$\begin{aligned} P(\omega^2) &= 0.7 - 0.7\omega^2, \\ Q(\omega^2) &= 0.7 - \omega^2, \\ p(\omega^2) &= -\omega^2, \\ q(\omega^2) &= 0, \end{aligned}$$

and then

$$\begin{aligned} F(\omega) &= \omega^2(0.7\omega^2 - 0.7), \\ G(\omega) &= (0.7 - 0.7\omega^2)^2 + \omega^2(0.7 - \omega^2)^2, \\ H(\omega) &= \omega^2(0.7 - \omega^2). \end{aligned}$$

it is not hard to see that $K^- = \left\{F(\sqrt{0.7}) = -0.147\right\}$ and $K^+ = \{0^+\}$. Therefore, by theorem (4.2) and lemma (4.3) the maximal interval of UDS-I is just described by $k^+_{\max} = \min\left\{-\frac{G(\sqrt{0.7})}{F(\sqrt{0.7})} = 0.3\right\} = 0.3$ and $k^-_{\min} = -\infty$. For the computing of the maximal UDS-II interval: For the polynomials $p_{A_{II}}(t) = t^3 + 0.7t^2 + 0.7t - 6.3$ and the polynomial $p_1(t) = t^2$ we have that

$$\begin{aligned} p_{A_{II}}(i\omega) &= (-6.3 - 0.7\omega^2) + i\omega(0.7 - \omega^2), \\ p_1(i\omega) &= -\omega^2, \end{aligned}$$

for which

$$\begin{aligned} P(\omega^2) &= -6.3 - 0.7\omega^2, \\ Q(\omega^2) &= 0.7 - \omega^2, \\ p(\omega^2) &= -\omega^2, \\ q(\omega^2) &= 0, \end{aligned}$$

and then

$$\begin{aligned} F(\omega) &= \omega^2(6.3 + 0.7\omega^2), \\ G(\omega) &= (6.3 + 0.7\omega^2)^2 + \omega^2(0.7 - \omega^2)^2, \\ H(\omega) &= \omega^2(0.7 - \omega^2). \end{aligned}$$

we have that $K^- = \{0^-\}$ and $K^+ = \left\{F(\sqrt{0.7}) = 4.753\right\}$. Therefore, by theorem (4.2) and lemma (4.3) the maximal interval of UDS-II is just described by $k^+_{\max} = +\infty$ and $k^-_{\min} = \max\left\{-\frac{G(\sqrt{0.7})}{F(\sqrt{0.7})} = -9.7\right\} = -9.7$. Then, $k \in K_{\max} = (-9.7, 0.3)$.

If $k = 0$ and $x_0 = (0.0569, 0.02847, 0.09492)$ the system (18) has multiscroll attractor as shown in Figure 2.

For $k = 0.2$ and $x_0 = (0.0569, 0.02847, 0.09492)$ the system (18) has a multiscroll attractor as shown in Figure 3.

## 6 Conclusions

In this paper, we have introduced a family of multi-scroll attractors generated by means of a family of switching systems with a control comprises by a multi-saturated signal and a feedback control with gain parameter $k$. We provide necessary and sufficient conditions to preserves multi-scroll attractors. We have also introduced the concept of the maximal robust dynamics interval for the control gain parameter $k$. An example was given to show how the developed theory works.

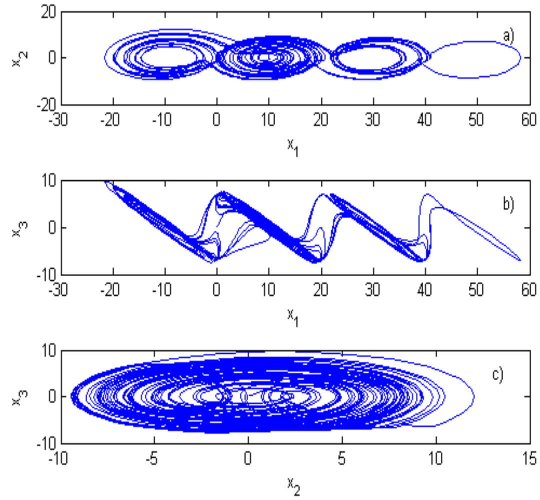



Figure 2: The projection of the attractor onto the planes: a) $(x_1, x_2)$; b) $(x_1, x_3)$; and c) $(x_2, x_3)$ for $k = 0$.

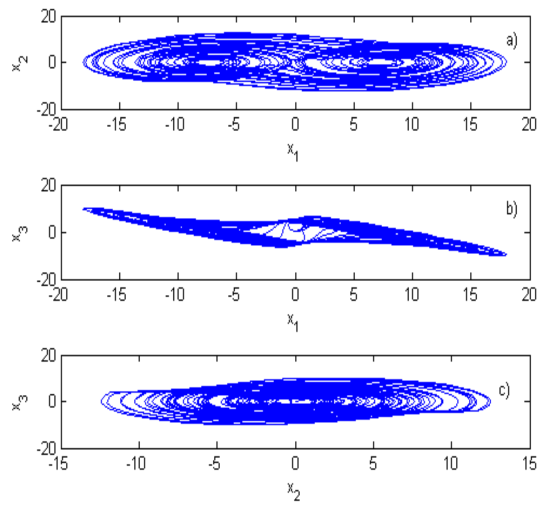



Figure 3: The projection of the attractor onto the planes: a) $(x_1, x_2)$; b) $(x_1, x_3)$; and c) $(x_2, x_3)$ for $k = 0.2$.

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## References

[1] B. Aguirre-Hernández, E. Campos-Cantón, J. A. López-Rentería & E. C. Díaz-González, *A polynomial approach for generating a monoparametric family of chaotic attractors via switched linear systems*, Chaos, Solitons & Fractals, **71**, pp. 100-106, (2015).

[2] B. Aguirre-Hernández, C. Ibarra & R. Suárez, *Sufficient algebraic conditions for stability of cones of polynomials*, Systems Control Lett.,**46** (4) pp. 255–263 (2002).

[3] B. Aguirre-Hernández & R. Suárez, *Algebraic test for the Hurwitz stability of a given segment of polynomials*, Bol. Soc. Mat. Mexicana (3), **12**, pp. 261–275 (2006).

[4] Barmish B. R., *New tools for robustness of linear systems.*, Macmillan. xvi, New York, NY, 1994.

[5] S. P. Bhattacharyya, H. Chapellat & L. H. Keel, *Robust control: The parametric approach*, Prentice Hall, 1995.

[6] E. Campos-Cantón, J. G. Barajas-Ramírez, G. Solís-Perales & R. Femat,, *Multiscroll attractors by switching systems*, Chaos, **20**, 013116 (2010).

[7] E. Campos-Cantón, R. Femat & G. Chen, *Attractors generated from switching unstable dissipative systems*, Chaos, **22**, 033121 (2012).

[8] J. A. López-Renteria, B. Aguirre-Hernández & F. Verduzco, *The boundary crossing theorem and the maximal stability interval*, Math. Prob. in Engineering, **2011**, 123403 (2011).

[9] J. Lü, F. Han, X. Yu & G. Chen, *Generating* $3-D$ *multis-croll chaotic attractors: A hysteresis series switching method*, Automatica, **40** (10) pp. 1677–1687 (2004).

[10] J. Lü, G. Chen, X. Yu & H. Leung H., *Design and analysis of multiscroll chaotic attractors from saturated function series*, IEEE Transactions on Circuits and Systems, Part I, **51** (12) pp. 2476-2490 (2004).

[11] J. Lü, K. Murali, S. Sinha, H. Leung & M. A. Aziz-Alaoiu, *Generating multi-scroll chaotic attractors by thresholding*, Phys. Lett. A, **372** (18) 3234–3239 (2008).

[12] L. J. Ontanon-Garcia, E. Jimenez-Lopez, E. Campos-Canton, & M. Basin, *A family of hyperchaotic multi-scroll attractors in* $R^n$, Applied Mathematics and Computation **233**, pp. 522-533 (2014).

[13] C. Sánchez-López, R. Trejo-Guerra, J. M. Muñoz-Pacheco & E. Tlelo-Cuautle, *N-scroll chaotic attractors from saturated function series employing CCII+s*, Nonlinear Dynamics, **61**, pp. 331-341 (2010).

[14] J. A. K. Suykens & J. Vandewalle, *Generation of n-double scrolls (n=1;2;3;4;. . .)*, IEEE Transactions on Circuits and Systems, Part I, **40** (11), pp. 861–867 (1993).

[15] J. A. K. Suykens, A. Huang & L. O. Chua, *A family of n-scroll attractors from a generalized Chua's circuit*, International Journal of Electronics and Communications, **51** (3), pp. 131–138 (1997).

[16] Tang K. S., Zhong G. Q., Chen G. R. & Man K. F., *Generation of n-scroll attractors via sine function*, IEEE Transactions on Circuits and Systems, Part I, **48** (11), pp. 1369-1372 (2001).

[17] M. E. Yalcin, J. A. K. Suykens, J. Vandewalle and S. Özoğuz, *Families of scroll grid attractors*, Int. J. of Bifurcations and Chaos, **12** (1) pp. 23–41 (2002).

[18] S. Yu, J. Lü, H. Leung & G. Chen, *Design and implementation of n-scroll chaotic attractors from a general jerk circuit*, IEEE Transactions on Circuits and Systems I, **52** (7) pp. 1459–1476 (2005).