The following article appeared in *IFAC-PapersOnLine 51(33): 156-161 (2018);* and may be found at: https://doi.org/10.1016/j.ifacol.2018.12.110

Available online at www.sciencedirect.com

ScienceDirect

IFAC PapersOnLine 51-33 (2018) 156–161

A class of Chua-like systems with only two saddle-foci of different type

R.J. Escalante-González, * **H. E. Gilardi-Velázquez** ** **and E. Campos-Cantón** *

* *División de Matemáticas Aplicadas, Instituto Potosino de Investigación Científica y Tecnológica A.C. Camino a la Presa San José 2055 col. Lomas 4a Sección, 78216, San Luis Potosí, SLP, México Email:rodolfo.escalante@ipicyt.edu.mx, eric.campos@ipicyt.edu.mx*

** *Facultad de Ciencias, Universidad Autónoma de San Luis Potosí. Alvaro Obregón 64, 78000 San Luis Potosí, SLP, México Email:hector.gilardi@uaslp.mx*

Abstract: Since the reported Chua's system, several generalizations of this system have been presented, these approaches include new equilibria in order to obtain three or more scrolls in the attractor. One of these generalizations requires at least the same number of saddle-foci with local two-dimensional unstable manifolds as the desired number of scrolls. In this work, we present the generation of a double-scroll chaotic attractor called Chua-like system. Once that an equilibrium point has been removed from the Chua's system and there are only two saddle-foci of different class, i.e. the dimension of one of the local unstable manifolds is one while the other is of dimension two. The new class is constructed based on the existence of a heteroclinic loop by linear affine systems with two saddle-focus equilibrium points of different type. Furthermore, the chaotic behavior of the proposed system is tested by the maximum Lyapunov exponent and the $0-1$ chaos test.

Keywords: Chua-like system, piecewise linear systems, heteroclinic loop, chaotic behavior, complex systems.

1. INTRODUCTION

Chaos has been an extremely studied area in the last decades, and designing systems with chaotic behavior is of great interest for the scientific community. One of the most remarkable properties is that simpler nonlinear deterministic equations can have unpredictable (chaotic) long-term solution.

A chaotic dynamical system can be identified by the observation of an strange attractor in the phase-space, or by a Poincaré section in the phase space, whatever, there are analytic criteria to show and argue chaotic behavior in dynamical systems. These are for example; Lyapunov exponents Wolf et al. (1985), which show the trajectories divergence; bifurcations theory, in which can be seen how a dynamical system pass from a regular behavior to a chaotic one by a series of well known bifurcations, an example of this it is shown in Lorenz (1963) where chaotic behavior is seen through Hopf bifurcations; the 1-0 test Gottwald and Melbourne (2009) which based on the growth rate of the mean square displacement of a two-dimensional system driven by a time series from the system under test, allow us to distinguish between regular and chaotic dynamics; the Shilnikov method Silva (1993a), which allows to prove the existence of chaotic behavior via homoclinic or heteroclinic loops in Smale horse-shoe shape.

Some ways to construct chaotic systems have been reported based on piecewise linear systems and the existence of homoclinic and heteroclinic loops Li and Chen (2009). These piecewise linear systems are chosen in a way that the union between eigen-spaces is guaranteed, and accordingly, the existence of homoclinic or heteroclinic loops. In Wang and Yang (2017) and Wu et al. (2016) a design of chaos generator with a class of three-dimensional two-zone piecewise affine systems based on Shilnikov method is studied, a theory is established to guarantee the existence of a heteroclinic cycle connecting two saddle-focus equilibria of the same type, specifically the saddle-focus has a one dimensional unstable manifold and two dimensional stable manifold. It is worth noting that the case in which the saddle-focus equilibria are of different type is not considered.

In the literature there are many implemented methods to generate scroll attractors. The multi-scroll strange attractors result from the combination of several unstable "one spiral" trajectories by means of a switching given by the control law. Without loss of generality, we deal with a mechanism based on piecewise linear systems (PWL) in R^3 and a switching control law to generate PWL systems that produce multi-scroll attractors. This class of systems is constructed with unstable dissipative systems (UDS) and a control law to display various multi-scroll strange

Peer review under responsibility of International Federation of Automatic Control.
10.1016/j.ifacol.2018.12.110

attractors. So the idea has been to add equilibrium points to the Chua's system to generate multiscroll attractos. The inverse process is to avoid an equilibrium point of the Chua's system but a natural question arises about whether it is possible to maintain a double-scroll attractor.

In this work, we present a generalized theory which is capable of explaining different approaches to construct double-scroll strange attractors based on the use of two different types of saddle-focus equilibrium points (i.e. taking into account different stability of each equilibrium point) and step functions in R^3. The piecewise linear systems are designed in a way that the union between eigenspaces is guaranteed forming a heteroclinic loop having the equilibrium points as the initial and final points.

This paper is organized as follows: In Section. 2, some theory of piecewise linear systems and its relationship with chaos is introduced. In Section 3, a class of systems with two saddle-focus equilibrium points of different type which present a heteroclinic orbit is discussed. Numerical results of a particular system with a chaotic attractor are given. Finally, in Section 4, some conclusions are drawn.

2. HYPERBOLIC SETS AND PIECEWISE LINEAR SYSTEMS

Saddle equilibrium points, which connect to a stable manifold W^S and an unstable manifold W^U, are responsible for successive stretching and folding therefore play an important role in generating chaos. The stretching causes the system trajectories to exhibit sensitive dependence on initial conditions whereas the folding creates the complicated microstructure Devaney et al. (1993). The saddle points of a chaotic system in R^3 can be characterized into two types according to its eigenvalues $\Lambda = \{\lambda_1, \lambda_2, \lambda_3\} \in C$ (i) The saddle points that are stable in one of its components but unstable and oscillatory in the other two Campos-Cantón et al. (2010). That is, the stable component is corresponding to a negative real eigenvalue; i.e., $Re\{\lambda_1\} < 0$, $Im\{\lambda_1\} = 0$, whereas the unstable components are related with two complex conjugate eigenvalues; i.e., $Re\{\lambda_{2,3}\} > 0$, $Im\{\lambda_{2,3}\} \neq 0$. (ii) The saddle points that are stable and oscillatory in two of its components but unstable in the another one. That is, the dissipative components are oscillatory: $Im\{\lambda_{2,3}\} \neq 0$ and $Re\{\lambda_{2,3}\} < 0$, while the unstable component corresponds to the real positive eigenvalue $Re\{\lambda_1\} > 0$, $Im\{\lambda_1\} = 0$.

In general, hyperbolic chaotic dynamical systems in R^3 are related to the two types of UDS around equilibrium; for instance, Chua's system Matsumoto (1984) has two UDS Type I equilibria, symmetrically distributed, and another UDS Type II at the origin. Rössler's system Rössler (1976) can also be characterized through UDS Type I and Type II, and similarly some other systems CHEN and UETA (1999); Lorenz (1963); Campos-Cantón et al. (2008). A characteristic of all these systems is that their scrolls are generated from UDS Type I.

It is important emphasize that the double-scroll attractor displayed by Chua system has three equilibrium points and does not display a scroll around the equilibrium point at the origin. This is because at least two saddle-focus equilibrium points of the same class are needed with local unstable manifolds of dimension two (the complex conjugate roots of $\mathcal{P}(Df(x^*))$ have positive real part, where $\mathcal{P}(\cdot)$ denotes the characteristic polynomial of an operator and $Df(x^*)$ denotes the Jacobian matrix of f evaluated at the equilibrium pint x^*) and the equilibrium point at the origin has a stable manifold of dimension two.

Consider the system with an associated vector field of the form:

$$\dot{x} = Ax + B, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $A \in \mathbb{R}^{n\times n}$ is a linear operator and $B \in \mathbb{R}^n$ is a constant vector. The system (1) is called a linear affine system if $B \neq 0$ and linear system if $B = 0$. If A^{-1} exists then the equilibrium point of the system (1) which satisfies $\dot{x} = 0$ is given by $x^* = -A^{-1}B$. Furthermore the roots λ_i with $i = 1, \dots, n$ of the associated polynomial of the operator A, $\mathcal{P}(A)$ (commonly called eigenvalues of A) are different from zero. This implies there is no central manifold $W^c_{x^*}$, only a stable manifold $W^S_{x^*}$, an unstable manifold $W^U_{x^*}$ or both. In the last case the equilibrium point x^* is called a saddle.

A system with an associated vector field of the form $\dot{x} = f(x)$ is called piecewise linear (PWL) if $f(x)$ is as follows:

$$f(x) = \begin{cases} A_1x + B_1, & \text{if } x \in D_1; \\ A_2x + B_2, & \text{if } x \in D_2; \\ \vdots & \vdots \ \vdots \\ A_mx + B_m, & \text{if } x \in D_m; \end{cases} \tag{2}$$

where D_i with $i = 1, \dots, m$ are hyperbolic set of a partition of the phase space $\mathbb{R}^n$. Each hyperbolic set serves as a domain for the linear or linear affine subsystems such that $\bigcup_{i=1}^m D_i = \mathbb{R}^n$ and $\bigcap_{i=1}^m D_i = \emptyset$. Any x^* satisfying $\dot{x} = 0$ is called an equilibrium point.

There exist systems with chaotic attractors whose main behavioral mechanism has been explained through the presence of homoclinic orbits or heteroclinic loops for some selection of parameters.

A *homoclinic orbit* is defined as a bounded dynamical trajectory of the system that is *doubly asymptotic* to an equilibrium point Silva (1993b). A *heteroclinic orbit* is similar except that there are two distinct saddle foci being connected, one corresponding to the forward asymptotic time limit Silva (1993b). A *heteroclinic loop* is formed by the union of two or more heteroclinic orbits Silva (1993b).

For the proposed class of systems, a combination of two saddle-focus equilibrium points of different class are considered. The idea behind the construction is based on the presence of a heteroclinic loop, however the analysis of that loop is out of the scope of this work.

3. ATTRACTORS GENERATED BY THE TWO TYPES OF SADDLE-FOCUS EQUILIBRIUM POINTS

Consider the piecewise linear (PWL) system with an associated vector field of the form:

$$\dot{x} = \begin{cases} A_1x + B_1, \text{ si } s \leq 0; \\ A_2x + B_2, \text{ si } s > 0; \end{cases} \tag{3}$$

where $x = [x_1, x_2, x_3]^T \in \mathbb{R}^3$ is the state vector, $B_1, B_2 \in \mathbb{R}^3$ are real constant vectors, $s = x_1 + x_2$ defines the switching plane $S = \{x \in \mathbb{R}^3 : x_1 + x_2 = 0\}$ and

$A_1, A_2 \in \mathbb{R}^{3\times 3}$ are linear operators whose associated polynomials $\mathcal{P}(A_1)$ and $\mathcal{P}(A_2)$ present a non-zero real root and two complex conjugate roots whose real part has the opposite sign of the real one. The real root of $\mathcal{P}(A_1)$ has also the opposite sign of the real root of $\mathcal{P}(A_2)$. Thus each subsystem of the form $\dot{x} = A_i x + B_i$ has a saddle-focus equilibrium point of different class. With an appropriate selection of subsystems it is possible to generate a chaotic double scroll attractor.

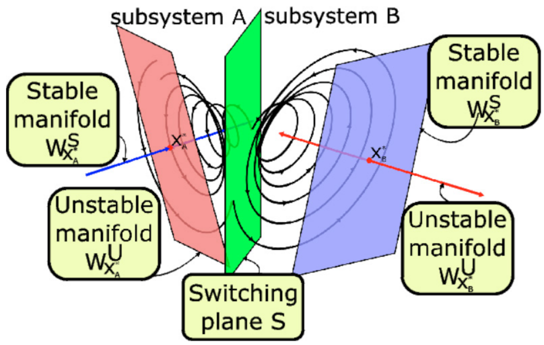

Fig. 1. Generation of the attractor by a PWL system.

In Figure 1 there is an illustration of the mechanism responsible for the attractor existence. The system is piecewise linear, so there is a subsystem assigned to each subset of the phase space, for this case the phase space is divided by a plane S that generates two subsets. In one of the subsets there is a saddle-focus equilibrium point x^*_A whose local stable manifold $W^S_{x^*_A}$ is one dimensional and its local unstable manifold $W^U_{x^*_A}$ is of dimension two. Since $W^U_{x^*_A}$ is a plane not parallel to the switching plane S any trajectory in that subset close to x^*_A will eventually go through the switching plane in a point close to or in $S \bigcap W^U_{x^*_A}$. In the other subset of the phase space there is the equilibrium point x^*_B whose local stable manifold $W^S_{x^*_B}$ is two dimensional and its local unstable manifold $W^U_{x^*_B}$ is of dimension one. The manifold $W^U_{x^*_B} = \text{span}\{u\}$ for a vector $u \in \mathbb{R}^3$ and $u \notin S$, furthermore $W^S_{x^*_B}$ divides that subset of the phase space in two subsets, in one of them the unstable manifold directs the flow against S. Then any trajectory crossing the plane S in that subset will return to S. The orientations of the local manifolds will determine the geometry of the attractor once the trajectories are bounded in this oscillating process around S.

In Figure 2 three qualitative trajectories are shown. The initial condition for the trajectories $x_0 \in W^U_{x^*_A}$ is a point close to the equilibrium point x^*_A. The first trajectory shown in Figure 2(a) is the case where the trajectory goes through the point $x_c \in W^U_{x^*_A} \bigcap S$ and then enters into a region where the orbit diverges not forming an attractor. A second case shown in Figure 2(b), is when $W^U_{x^*_A} \bigcap cl(W^S_{x^*_B}) \neq \emptyset$ where $cl(\cdot)$ denotes the closure, the trajectory goes through the point $x_c \in W^U_{x^*_A} \bigcap cl(W^S_{x^*_B})$ and tends to x^*_B as $t \to \infty$, thus an heteroclinic loop is formed. In the same way $cl(W^U_{x^*_B}) \bigcap W^S_{x^*_A} \neq \emptyset$ and then any trajectory starting in $W^U_{x^*_B}$ goes to x^*_A as $t \to \infty$ which form a second heteroclinic loop and a heteroclinic orbit is completed. Note that if $W^U_{x^*_A} \bigcap cl(W^S_{x^*_B}) \bigcap S \neq \emptyset$ and $W^S_{x^*_A} \bigcap cl(W^U_{x^*_B}) \bigcap S \neq \emptyset$ is guaranteed then there exists a heteroclinic orbit. A third case is shown in Figure 2(c) where the trajectory goes through $W^U_{x^*_A} \bigcap S$ and enters into a region where the equilibrium point x^*_B repels the trajectory against S which traps the trajectory between the two saddle equilibrium points and generates a chaotic attractor.

In Figure 3 the three trajectories cases aforementioned, respectively, are presented from another perspective view and the two-dimensional manifolds are represented as a line that corresponds to a vector of the two-dimensional manifold. This representation allows us to easily see the role of the manifolds in the behavior of the system close to the equilibria.

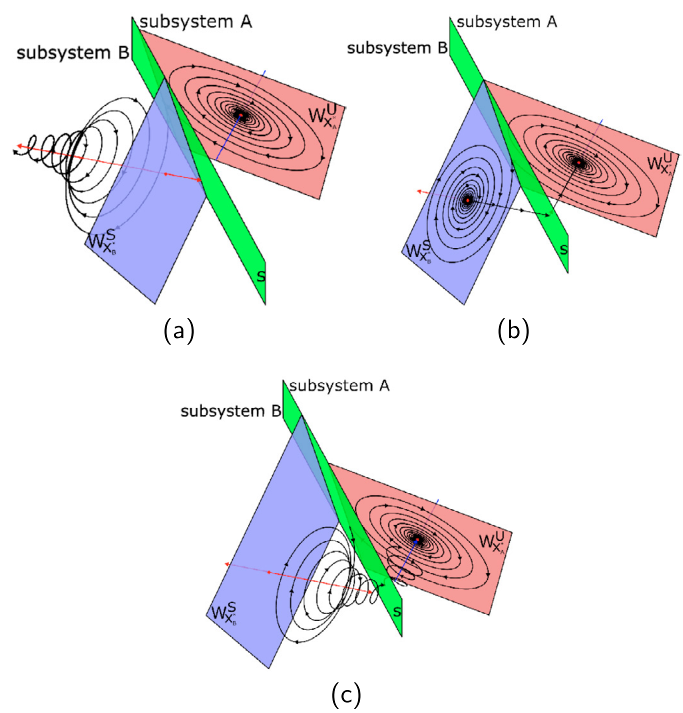

Fig. 2. Three qualitative trajectories in the PWL system with two different saddle focus equilibrium points.(a)Trajectory is trapped between the two equilibrium points. (b)Heteroclinic orbits are generated. (c)Trajectory escapes and gets away the surface S.

The interest is to form a chaotic attractor as is depicted in third case aforementioned. The matrix A_1 has been choosen in controllable canonical form due to its simplicity, the matrix A_2 presents an apparent more complicated form, however, its form provide a useful location of the manifolds and the eigenvalues for the proposed construction, once that the matrix A_i are established and according to the their manifolds the location of the equilibrium points are selected based on the aforementioned idea, then the vectors B_1 and B_2 are obtained by $-A_i x^* = B_i$. To illustrate a construction that exhibits this behavior as the third case consider the particular system of the form (3) with:

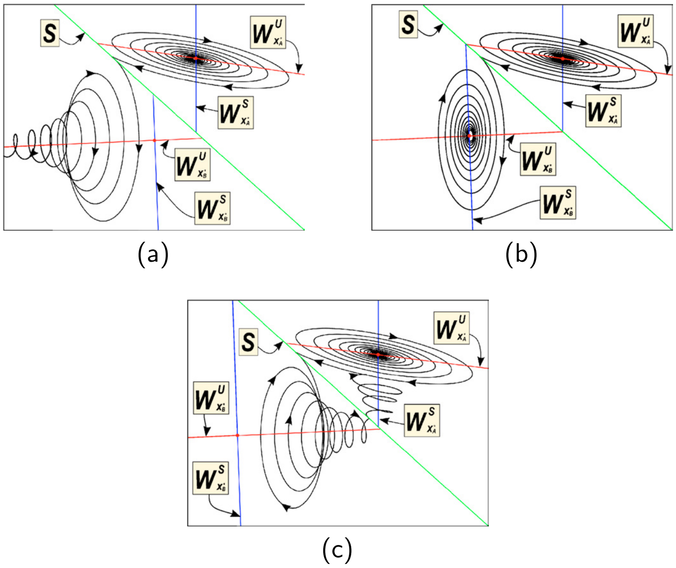

Fig. 3. Three qualitative trajectories of Figure 2 seen from another angle and where the two-dimensional manifolds are represented only by a line that correspond to a vector of the two-dimensional manifold.

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.313 & -6.25 & -0.15 \end{bmatrix}, \qquad (4)$$

$$A_2 = \begin{bmatrix} 0.9364 & 0.2609 & 1.3045 \\ -0.8548 & -0.1524 & -0.2619 \\ -5.1564 & -0.1168 & -0.6840 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1 \\ -1 \\ -5.474 \end{bmatrix}, \; B_2 = \begin{bmatrix} -2.3306 \\ 1.0869 \\ 4.9756 \end{bmatrix}. \qquad (5)$$

The roots of $\mathcal{P}(A_1)$ are $\lambda_1 = 0.05, \lambda_{2,3} = -0.1 \pm 2.5i$ and the roots of $\mathcal{P}(A_2)$ are $\lambda_1 = -0.1, \lambda_{2,3} = 0.1 \pm 2.5i$. The resulting double-scroll attractor is shown in Figure 4. It is worth mentioning that the double-scroll attractor generated by our Chua-like system differentiates from other Chua-like attractors where in order to get two scrolls, there is a need of at least two saddle-focus equilibrium points of the same class with local unstable manifolds of dimension two (the complex conjugate roots of $\mathcal{P}(Df(x^*))$ have positive real part). Figure 5 shows the time series of the system, which are a mixed underdamped and increasing oscillations signals which are in concordance with the two types of equilibrium points.

The basin of attraction estimated numerically is shown in Figure 6 with blue dots. It was obtained by evaluating the points in the grid given by $-10 \leq x_1 \leq 10$, $-10 \leq x_2 \leq 10$ and $-10 \leq x_3 \leq 10$ with an increment of $\Delta = 0.1$ in each direction, a point in the grid was considered part of the basin of attraction if $||x(1000)|| < 8$.

3.1 Chaos tests

In order to verify the chaotic behavior of the attractor the Maximum Lyapunov Exponent (MLE) was calculated based on the algorithm proposed by Rosenstein et al. (1993). A main trajectory along with ten additional trajectories with an orthogonal initial separation of $d_0 = 10^{-8}$ were

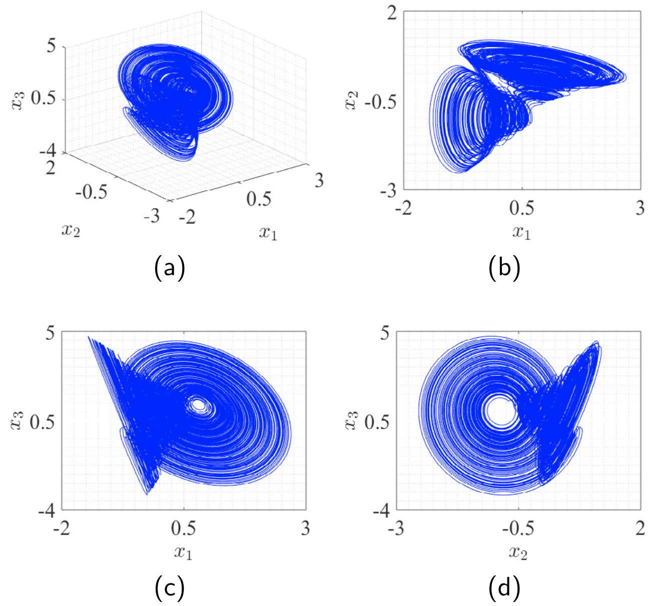

Fig. 4. Attractor of the system (3) with A_1, A_2, B_1 and B_2 for the initial condition $(0, 0, 0)$ $(t = 1000s)$ in (a) the space (x_1, x_2, x_3) and its projections onto the planes: (b) (x_1, x_2), (c) (x_1, x_3) and (d) (x_2, x_3).

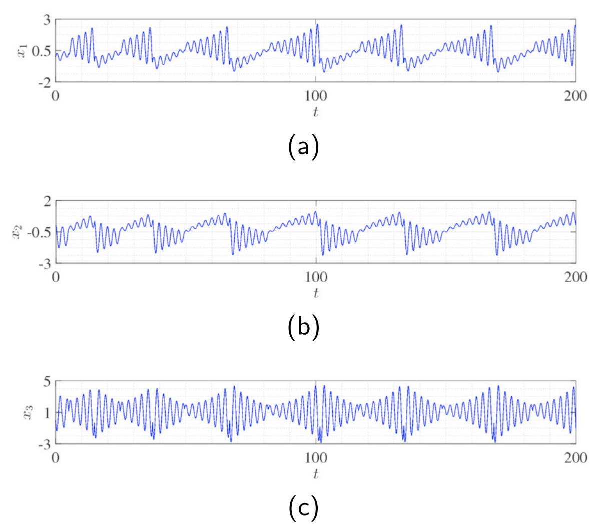

Fig. 5. Time series of the attractor shown in Figure 4. In (a) x_1 state, (b) x_2 state and (c) x_3 state.

used for the calculation. The trajectories were obtained using a fourth order Runge-Kutta with a step size of $h = 0.001$. The calculated value is MLE=0.077955 and the plot of the average separation versus the time used is shown in Figure 7.

The test for Chaos 0-1 proposed in Gottwald and Melbourne (2009) was also performed which allow us to distinguish chaotic dynamics. Rather than requiring phase space reconstruction which is necessary to apply standard Lyapunov exponent methods to the analysis of discretely sampled data, the test works directly with the time series and does not involve any preprocessing of the data. The test requires only a minimal computational effort independent

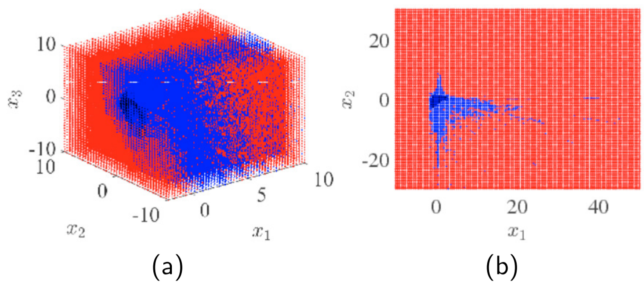

Fig. 6. Basin of attraction estimated numerically of the attractor in Figure 4 is marked with blue dots in the phase space (a)$x_1 - x_2 - x_3$ and (b) part of it at $x_3 = 0$.

of the dimension of the underlying dynamical system under investigation Gottwald and Melbourne, 2016.

The test requires a time series $\phi(n)$ which was conformed by sampling every $\tau = 0.4$ the x_1 coordinate of a trajectory of the attractor. The trajectory was obtained by Runge-Kutta with a step size of $h = 0.01$. The test consists in using the time series $\phi(n)$ to drive a proposed two dimensional system given in Gottwald and Melbourne (2009):

$$\begin{aligned} p(n+1) &= p(n) + \phi(n)\cos cn, \\ q(n+1) &= q(n) + \phi(n)\sin cn, \end{aligned} \tag{6}$$

with $c \in (0, 2\pi)$ fixed. Then the growth rate of the mean square displacement k_c is calculated for each value of c in a set. The median of the set of calculated k_c values is K, which distinguishes between chaotic ($K = 1$) and regular motion ($K = 0$). For the proposed system with a double scroll, the 1-0 test yields a result of $K = 0.9554$, which gives an indication of the existence of chaos. In Figure 8 are shown the values of K_c calculated in the test.

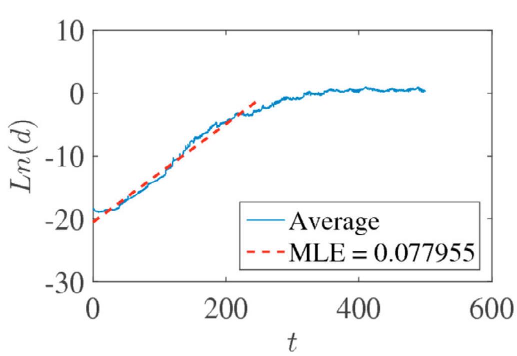

Fig. 7. Maximum Lyapunov exponent calculation.

4. CONCLUSION

In this work the question of whether it is possible or not to generate a chaotic double scroll attractor once an equilibrium point is removed from Chua's system and there are only two saddle-foci of different class was explored. A class of PWL system is reported as Chua-like system, the piecewise linear approach in the construction, based on the existence of an heteroclinic loop, allows us to explain the qualitative behavior based on the local equilibria and the commutation. The attractors obtained using the proposed construction differs from other reported approaches, in the sense that this does not require the existence of two saddle-focus equilibrium points of the same type, where the local unstable manifold is of dimension two. As well as the UDS character. Also the chaotic behavior was validated via the Maximum Lyapunov Exponent and the $0-1$ chaos test.

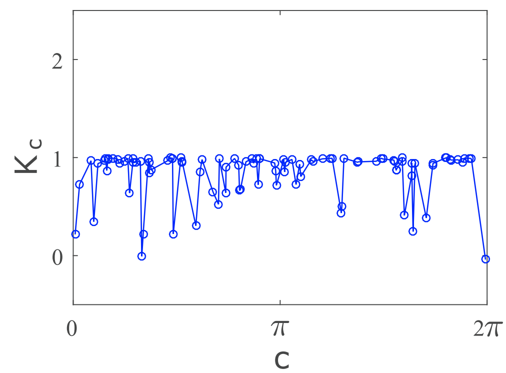

Fig. 8. Asymptotic growth rates K_c calculated for the double scroll attractor systems.

5. ACKNOWLEDGMENTS

R.J.Escalante-González and H.E.Gilardi-Velázquez students of control and dynamical systems at IPICYT, are thankful to CONACYT (México) for the scholarships granted.

REFERENCES

Campos-Cantón, E., Campos-Cantón, I., González Salas, J., and Cruz Ordaz, F. (2008). A parameterized family of single-double-triple-scroll chaotic oscillations. *Revista mexicana de física*, 54, 411 – 415.

Campos-Cantón, E., Barajas-Ramirez, J., Solís-Perales, G., and Femat, R. (2010). Multiscroll attractors by switching systems. *Chaos*, 20(1), 013116–1–6.

CHEN, G. and UETA, T. (1999). Yet another chaotic attractor. *International Journal of Bifurcation and Chaos*, 09(07), 1465–1466.

Devaney, R.L., Siegel, P.B., Mallinckrodt, A.J., and McKay, S. (1993). A first course in chaotic dynamical systems: Theory and experiment. *Computers in Physics*, 7(4), 416–417.

Gottwald, G.A. and Melbourne, I. (2009). The 0-1 test for chaos: A review. *SIAM J. Appl. Dyn. Syst.*, 8(1), 129–145.

Gottwald, G.A. and Melbourne, I. (2016). *The 0-1 Test for Chaos: A Review*, 221–247. Springer Berlin Heidelberg, Berlin, Heidelberg.

Li, G. and Chen, X. (2009). Constructing piecewise linear chaotic system based on the heteroclinic Shil'nikov theorem. *Communications in Nonlinear Science and Numerical Simulation*, 14(1), 194 – 203.

Lorenz, E.N. (1963). Deterministic nonperiodic flow. *Journal of the Atmospheric Sciences*, 20(2), 130–141.

Matsumoto, T. (1984). A chaotic attractor from chua's circuit. *IEEE Transactions on Circuits and Systems*, 31(12), 1055–1058.

Rosenstein, M., Collins, J., and de Luca, C. (1993). A practical method for calculating largest lyapunov exponents from small data sets. *Physica D*, 65, 117–134.

Rössler, O. (1976). An equation for continuous chaos. *Physics Letters A*, 57(5), 397 – 398.

Silva, C.P. (1993a). Shil'nikov's theorem-a tutorial. *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, 40(10), 675–682.

Silva, C.P. (1993b). Shil'nikovś theorem-a tutorial. *IEEE TRANSACTIONS ON CIRCUITS AND SYSTEMS—I: FUNDAMENTAL THEORY AND APPLICATIONS*, 40(10), 675–682.

Wang, L. and Yang, X.S. (2017). Heteroclinic cycles in a class of 3-dimensional piecewise affine systems. *Nonlinear Analysis: Hybrid Systems*, 23, 44 – 60.

Wolf, A., SWift, J., Swinney, H., and Vastano, J. (1985). Determining lyapunov exponents from a time series. *Physica D*, 16, 285–317.

Wu, T., Wang, L., and Yang, X.S. (2016). Chaos generator design with piecewise affine systems. *Nonlinear Dynamics*, 84(2), 817–832. doi:10.1007/s11071-015-2529-8.