Radius Evolution for Bubbles with Elastic Shells

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We present an analysis of an extended Rayleigh-Plesset (RP) equation for a three dimensional cell of a microorganism such as a bacteria or a virus in some liquid, where the cell membrane in bacteria or the envelope in viruses possess elastic properties. To account for rapid changes in the shape configuration of such microorganisms, the bubble membrane/envelope must be rigid to resist large pressures while being flexible to adapt to growth or decay. In this paper, such properties are embodied in the RP equation by including a pressure bending term that is proportional to the square of the curvature of the elastic wall. Analytical solutions to this extended equation are obtained.

I. INTRODUCTION

It is well established that the size evolution of unstable, spherical cavitation bubbles in 3-dimensions with surface tension is governed by the well-known RP equation [1–3]

$$\rho_w\Big(RR_{TT}+\frac{3}{2}{R_T}^2\Big)=\Delta P-\frac{2\sigma}{R}\ . \tag{1}$$

where ρ_w is the density of the water, $R(T)$ is the radius of the bubble, $\Delta P = p - P_\infty$ is the pressure drop between the uniform pressure inside the bubble and the external pressure in the liquid at infinity (hydrostatic and sound field for example), and σ is the surface tension of the bubble.

In the simpler form with only the pressure difference in the right hand side, Eq. (1) was first derived by Rayleigh in 1917 [1], but it was only in 1949 that Plesset developed the full form of the equation and applied it to the problem of traveling cavitation bubbles [2].

On the other hand, we can extend the RP equation to study the evolution of the cell wall of microorganisms such as bacteria and viruses, by the inclusion of an additional term that accounts for the bending pressure of the thin outer shell. However, the effects of mechanical properties of the outer shell in controlling and maintaining the sizes of microorganisms are not well known. Because the elastic energy per unit area of bending a thin shell is proportional to the square of the curvature [4], the extended RP equation (ERP) can be modified to include this additional bending pressure term $p_b = Yh^2/R^2$ of the thin outer shell of elastic modulus Y, and thickness h coating the cell to read

$$\rho_w\Big(RR_{TT}+\frac{3}{2}{R_T}^2\Big)=\Delta P-\frac{2\sigma}{R}+\frac{Yh^2}{R^2}\ . \tag{2}$$

Typical fixed values that we will use are $\rho_w = 10^3\ Kg/m^3$, $R_0 = 10^{-6}\ m$, $P_\infty = 101325\ Pa$, $h = 10^{-8}\ m$. The values for Young's modulus and surface tension σ are allowed to vary in the ranges $[1\times10^7, 5\times10^8]\ Pa$ and $[1\times10^{-4}, 2\times10^{-2}]\ N/m$.

In this paper, we find analytical solutions of the "bubbles with shell" model as expressed by Eq. (2), and we show that the bending pressure controls the sizes of bacteria and viruses.

II. INTEGRATING FACTOR AND INTEGRATION VIA WEIERSTRASS EQUATION

To solve (2) we will use two initial conditions $R(0) = R_0$, and $R_T(0) = 0$, and by introducing nondimensional variables given by $R = R_0u$, and $T = T_ct$, for a vacuous bubble of zero internal pressure $p = 0$, then (2) becomes

$$uu_{tt}+\frac{3}{2}{u_t}^2=\frac{{T_c}^2}{{R_0}^2\rho_w}\left(-P_\infty-\frac{2\sigma}{R_0u}+\frac{Yh^2}{{R_0}^2u^2}\right)\ , \tag{3}$$

subject to new initial conditions given by $u(0) = 1$ and $u_t(0) = 0$. The collapse (Rayleigh) time T_c of the vacuous bubble such as a cell, of 1 μm in radius is given by [5]

$$T_c = \frac{\Gamma(\frac{5}{6})}{\Gamma(\frac{4}{3})}\sqrt{\frac{\pi}{6}}R_0\sqrt{\frac{\rho_w}{P_\infty}} = 0.914681\ R_0\sqrt{\frac{\rho_w}{P_\infty}} = 0.914681 \times 9.934401 \times 10^{-8}\ sec. = 9.08681 \times 10^{-8}\ sec\ . \tag{4}$$

Furthermore, we let $\frac{T_c{}^2 P_\infty}{R_0{}^2 \rho_w} = \xi^2$, where $\xi = 0.914681$ is an universal constant known as the 3-dimensional Rayleigh factor [6], and make the notation $\frac{\xi^2}{P_\infty}\frac{2\sigma}{R_0} = \gamma$, and $\frac{\xi^2}{P_\infty}\frac{Yh^2}{R_0{}^2} = \alpha$ (also noticing that the quotients $\gamma/\xi^2 = p_\sigma/P_\infty$, and $\alpha/\xi^2 = p_\alpha/P_\infty$ are pressure quotients with surface pressure $p_\sigma = 2\sigma/R_0$ and bending pressure $p_\alpha = Yh^2/R_0{}^2$), we write (3) in the form

$$uu_{tt} + \frac{3}{2}u_t{}^2 = -\xi^2 - \frac{\gamma}{u} + \frac{\alpha}{u^2}\ . \tag{5}$$

For these values of ranges of parameters we obtain $\gamma \in [0.0016514, 0.330281]$, $\alpha \in [0.00825701, 0.412851]$, surface pressure $p_\sigma \in [2 \times 10^2, 4 \times 10^4]\ Pa$, and bending pressure $p_\alpha \in [10^3, 5 \times 10^4]\ Pa$.

By multiplying (5) by the integrating factor $2u^2u_t$, we have the conservation form

$$\frac{d}{dt}\left[u^3 u_t{}^2 + \frac{2\xi^2}{3}u^3 + \gamma u^2 - 2\alpha u\right] = 0\ , \tag{6}$$

so that

$$u_t{}^2 = -\frac{2\xi^2}{3} - \frac{\gamma}{u} + \frac{2\alpha}{u^2} + \frac{c_1}{u^3}, \tag{7}$$

where c_1 is an integration constant that varies linearly with respect to the surface tension σ and Young's modulus Y. Using the two initial conditions, this constant is

$$c_1(\alpha, \gamma) = -2\alpha + \gamma + \frac{2\xi^2}{3}. \tag{8}$$

For the empty cavity without surface tension or bending pressure, $\gamma = 0$ and $\alpha = 0$, and we have $c_1 = \frac{2\xi^2}{3}$, which reduces (7) to

$$u_t{}^2 = \frac{2\xi^2}{3}\left(\frac{1}{u^3} - 1\right). \tag{9}$$

The solution of this equation is found by inversion of the integral

$$t(u) = \frac{1}{\xi}\sqrt{\frac{3}{2}}\int_u^1 \frac{w^{3/2}dw}{\sqrt{1-w^3}}\ , \tag{10}$$

which in parametric form becomes

$$t(u) = \frac{2}{5\xi}\sqrt{\frac{3}{2}}\left[\frac{\sqrt{\pi}\Gamma\left(\frac{11}{6}\right)}{\Gamma\left(\frac{4}{3}\right)} - u^{5/2}\,{}_2F_1\left(\frac{1}{2}, \frac{5}{6}; \frac{11}{6}; u^3\right)\right]\ . \tag{11}$$

Notice that the collapse time is obtained from (10) by setting $u = 0$ which give the Rayleigh factor $\xi = \frac{\Gamma(\frac{5}{6})}{\Gamma(\frac{4}{3})}\sqrt{\frac{\pi}{6}}$.

The phase portrait of (9) and the parametric hypergeometric solution for the case $\gamma = 0$ and $\alpha = 0$ from (11) are displayed in Fig. 1.

The contributions of surface tension and bending pressure must be taken into account simultaneously in studying the deformations of microorganisms since the mechanical equilibrium is determined by he surface tension. Thus, the ratio of the two constants α and γ is a very important dimensionless parameter given by $q \equiv \frac{2\alpha}{\gamma} = \frac{Yh^2}{R_0\sigma} > 0$. When $q < 1$ then $\gamma > 2\alpha$ so we have the regime where $\sigma > \frac{Yh^2}{R_0}$. This is the regime of high surface tension which is

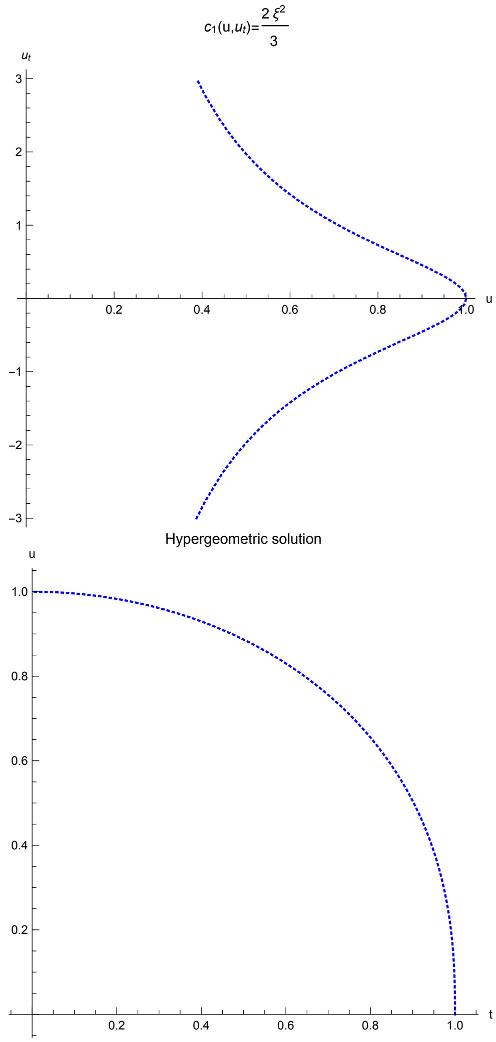

FIG. 1: The phase portrait of (9) and the corresponding parametric hypergeometric solution for the case $\gamma = 0$ and $\alpha = 0$, i.e., $(Y_0, \sigma_0) = (0, 0)$, from (11).

characterized by irreversible deformation and occurs when the stresses approach the elastic limit. This region is the plastic deformation region. On the other hand, when $q > 1$ then $\gamma < 2\alpha$ and we have the regime where $\sigma < \frac{Yh^2}{R_0}$ which is the region when microorganisms recover their shape after the external stress has been removed. This region is the elastic deformation region. At the boundary between the two regions, $q = 1$, there is the critical radius $R_c = \frac{Yh^2}{\sigma}$ which plays an important role in determining the size of the microorganisms. In each of the plastic and elastic regions analytical solutions will be found next.

The approach to integrate (7) is to transform it into an equation in which the right hand side is a polynomial in u. Namely, we will use the Sundman transformation $dt = u^2 d\tau$ where τ is the new independent variable which gives the Weierstrass elliptic equation

$$u_\tau{}^2 = -\frac{2\xi^2}{3}u^4 - \gamma u^3 + 2\alpha u^2 + c_1 u \equiv Q(u). \tag{12}$$

It is well known [7–9] that the solutions $u(\tau)$ of

$$u_\tau{}^2 = \mathrm{A}_4 u^4 + 4\mathrm{A}_3 u^3 + 6\mathrm{A}_2 u^2 + 4\mathrm{A}_1 u + \mathrm{A}_0, \tag{13}$$

can be expressed in terms of Weierstrass elliptic functions $\wp(\tau; g_2, g_3)$, which is a solution to

$$\wp_\tau{}^2 = 4\wp^3 - g_2\wp - g_3 \ , \tag{14}$$

via the transformation

$$u(\tau) = \hat{u} + \frac{\sqrt{Q(\hat{u})}\wp'(\tau+\tau_0; g_2, g_3) + \frac{1}{2}Q'(\hat{u})\Big[\wp(\tau+\tau_0; g_2, g_3) - \frac{1}{24}Q''(\hat{u})\Big] + \frac{1}{24}Q(\hat{u})Q^{(3)}(\hat{u})}{2\Big[\wp(\tau+\tau_0; g_2, g_3) - \frac{1}{24}Q''(\hat{u})\Big]^2 - \frac{1}{48}Q(\hat{u})Q^{(4)}(\hat{u})} \ , \tag{15}$$

where $\hat{u}$ can be taken not necessarily as a root of $Q(u)$, and g_2, g_3 are elliptic invariants of $\wp(\tau)$, given by

$$\begin{aligned} g_2 &= \mathrm{A}_4\mathrm{A}_0 - 4\mathrm{A}_3\mathrm{A}_1 + 3\mathrm{A}_2{}^2 = \frac{\alpha^2}{3} + \frac{c_1\gamma}{4} \ , \\ g_3 &= \mathrm{A}_4\mathrm{A}_2\mathrm{A}_0 + 2\mathrm{A}_3\mathrm{A}_2\mathrm{A}_1 - \mathrm{A}_4\mathrm{A}_1{}^2 - \mathrm{A}_2{}^3 - \mathrm{A}_3{}^2\mathrm{A}_0 = \frac{1}{216}\left(-8\alpha^3 + 9c_1{}^2\xi^2 - 9c_1\alpha\gamma\right) \ . \end{aligned} \tag{16}$$

These invariants are components of the modular discriminant

$$\Delta = g_2{}^3 - 27g_3{}^2 = \frac{c_1{}^2}{192}\left(16\alpha^3\xi^2 + 3\alpha^2\gamma^2 - 9c_1{}^2\xi^4 + 18c_1\alpha\xi^2\gamma + 3c_1\gamma^3\right) \tag{17}$$

and together are used to classify the solutions of (12). In particular, choosing $\hat{u} = 0$, which is a root of $Q(u) = 0$, then the general solution (15) takes the much simpler form

$$u(\tau) = \frac{Q_u(0)}{4\wp(\tau+\tau_0; g_2, g_3) - \frac{Q_{uu}(0)}{6}} = \frac{A_1}{\wp(\tau+\tau_0; g_2, g_3) - \frac{A_2}{2}} \ . \tag{18}$$

This solution can also be explained by letting $u(\tau) = \frac{1}{v(\tau)}$ in (13) which gives the Weierstrass equation

$$v_\tau{}^2 = A_4 + 4A_3v + 6A_2v^2 + 4A_1v^3 \ , \tag{19}$$

which is

$$v_\tau{}^2 = -\frac{2\xi^2}{3} - \gamma v + 2\alpha v^2 + \left(-2\alpha + \gamma + \frac{2\xi^2}{3}\right)v^3 \ . \tag{20}$$

The standard form of (19) given by (14) can be found for $A_1 \neq 0$ by the linear transformation

$$v(\tau) = \frac{1}{A_1}\left(\wp(\tau; g_2, g_3) - \frac{A_2}{2}\right) \tag{21}$$

yielding (18). Using the initial conditions together with (8), the constant τ_0 can be found numerically from the equation

$$\wp(\tau_0; g_2, g_3) = \frac{3\gamma + 2\xi^2 - 4\alpha}{12} \ , \tag{22}$$

and thus the general solution to (12) in parametric form is

$$\begin{aligned} u(\tau) &= \frac{3\alpha - \xi^2 - \frac{3\gamma}{2}}{\alpha - 6\wp\left[\tau+\tau_0; \frac{\alpha^2}{3} + \frac{\gamma}{4}\left(-2\alpha + \frac{2\xi^2}{3} + \gamma\right), \frac{1}{216}\Big(-8\alpha^3 + 3\gamma(6\alpha - 2\xi^2 - 3\gamma)\alpha + \xi^2(-6\alpha + 2\xi^2 + 3\gamma)^2\Big)\right]} \\ t(\tau) &= \int_0^\tau u^2(\zeta)d\zeta \ . \end{aligned} \tag{23}$$

For our set of minimum, maximum, extreme zero values, and average values of Young's modulus and surface tension we obtain the following analytic solutions

$$
\begin{aligned}
u_m(\tau) &= \frac{0.135725}{\wp(\tau+3.28025;0.000246862,0.0102743)-0.00137617} \\
u_M(\tau) &= \frac{0.0155851}{\wp(\tau+5.28075;0.0619627,-0.00282496)-0.0688085} \\
u_0(\tau) &= \frac{\pi\Gamma\left(\frac{5}{6}\right)^2}{36\Gamma\left(\frac{4}{3}\right)^2\wp\left(\tau+3.25193;0,\frac{\pi^3\Gamma\left(\frac{5}{6}\right)^6}{11664\Gamma\left(\frac{4}{3}\right)^6}\right)} \\
\bar{u}(\tau) &= \frac{0.13944}{\wp(\tau+3.15848;0.0254377,0.0105037)-0.0138305}\ .
\end{aligned}
\tag{24}
$$

Phase portraits of the elliptic Weierstrass equation (12) for constant c_1 and the corresponding solutions of (24) are displayed in Fig. 2 for the numerical values of the parameters presented in Table I.

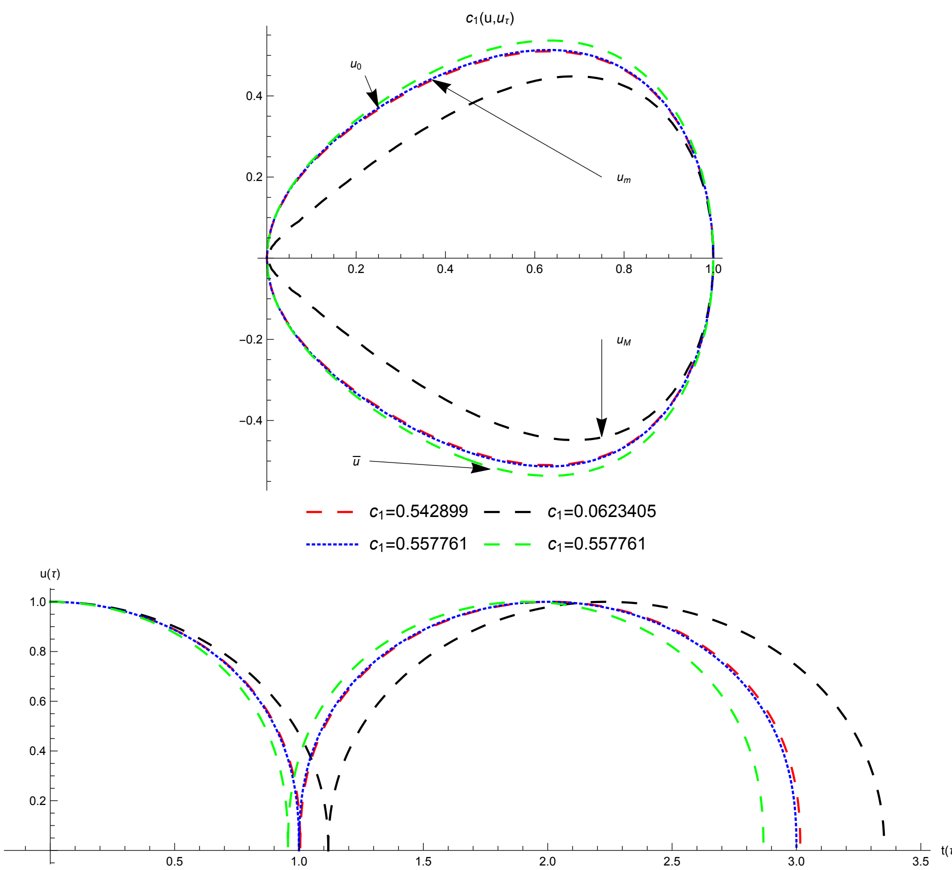

FIG. 2: The phase portrait of (12) which indicates periodic solutions given by (24). $c_1 = 0.542899$ and $c_1 = 0.0623405$ correspond to minimum (Y_m, σ_m) and maximum (Y_M, σ_M) values, respectively. The two special cases of $c_1 = 0.557761$ correspond to $(Y_0, \sigma_0) = (0, 0)$, and $(\bar{Y}, \bar{\sigma})$. For $(Y_0, \sigma_0) = (0, 0)$, the hypergeometric solution can be parameterized in terms of the $\wp$ elliptic function given by $u_0(\tau)$.

TABLE I: The numerical values of the parameters used in the phase portraits depicted in Fig. 2

values for general solution	Y $[Pa]$	σ $[N/m]$	R_c $[m]$	γ	α	c_1	q	u
minimum	10^7	10^{-4}	10^{-5}	0.0016514	0.00825701	0.542899	10	u_m
maximum	5×10^8	2×10^{-2}	2.5×10^{-6}	0.330281	0.412851	0.0623405	2.5	u_M
special I (extreme)	0	0	undefined	0	0	0.557761	undefined	u_0
special II (mean)	1.005×10^8	1.005×10^{-2}	10^{-6}	0.165966	0.0082893	0.557761	1	$\bar{u}$

A. Cnoidal solutions

This type of periodic solutions is obtained for the lemniscatic case $g_3 = 0$ which gives $\alpha = \frac{1}{4}(2\xi^2 + 3\gamma)$ and is equivalent to $q = \frac{3}{2} + \frac{\xi^2}{\gamma}$. In this case, $c_1 = -\frac{1}{6}(2\xi^2 + 3\gamma)$, $g_2 = \frac{1}{48}(2\xi^2 + \gamma)(2\xi^2 + 3\gamma)$, the roots of $Q(v)$ are real and (19) can be factored as

$$ {v_\tau}^2 = -\frac{1}{6}(v-1)\left[-4\xi^2 + v(2\xi^2 + 3\gamma)(v-2)\right] \ . \tag{25} $$

These real roots are

$$ e_3 = 1 - \frac{2\xi^2 + \gamma}{\sqrt{\frac{4\xi^4}{3} + \frac{8\xi^2\gamma}{3} + \gamma^2}} \ , \quad e_2 = 1 \ , \quad e_1 = 1 + \frac{2\xi^2 + \gamma}{\sqrt{\frac{4\xi^4}{3} + \frac{8\xi^2\gamma}{3} + \gamma^2}} \ , $$

and although the Weierstrass unbounded function given by (14) has poles aligned on the real axis of the $\tau - \tau_0$ complex plane, we can choose τ_0 in such a way to shift these poles a half of period above the real axis, so that the $\wp$ elliptic function reduces to the Jacobi elliptic function given by

$$ \wp(\tau; g_2, 0) = e_3 + (e_2 - e_3)\mathrm{sn}^2[\sqrt{e_1 - e_3}(\tau + \tau_0); m] = -\frac{\sqrt{g_2}}{2}\mathrm{cn}^2\left[\sqrt[4]{g_2}(\tau + \tau_0); \frac{1}{\sqrt{2}}\right] \ , \tag{26} $$

with elliptic modulus $m = \sqrt{\frac{e_2 - e_3}{e_1 - e_3}}$. Thus, the solutions (21) reduces to

$$ v(\tau) = 1 - \frac{24}{\xi^2 + 3\gamma}\,\wp(\tau; g_2, 0) \ . \tag{27} $$

For the lemniscatic case, this solution is reduced using the transformation (26) to cnoidal waves, and it becomes

$$ v(\tau) = 1 + \frac{2\xi^2 + \gamma}{\sqrt{\frac{4\xi^4}{3} + \frac{8\xi^2\gamma}{3} + \gamma^2}}\mathrm{cn}^2\left[\frac{\sqrt[4]{(2\xi^2 + \gamma)(2\xi^2 + 3\gamma)}}{2\sqrt[4]{3}}(\tau + \tau_0); \frac{1}{\sqrt{2}}\right] \ . \tag{28} $$

To satisfy the initial condition, τ_0 is found numerically from

$$ \mathrm{cn}\left[\frac{\sqrt[4]{(2\xi^2 + \gamma)(2\xi^2 + 3\gamma)}}{2\sqrt[4]{3}}\tau_0; \frac{1}{\sqrt{2}}\right] = 0 \ . \tag{29} $$

Choosing the realistic numerical values $Y = 6.57375 \times 10^8$ Pa, $\sigma = 1.005 \times 10^{-2}$ N/m, and $R_c = 6.54104 \times 10^{-6}$ m, one can obtain the values of γ, α, c_1, and q as 0.165966, 0.542795, -0.361864, 6.54104, respectively. The resulting analytic solution is

$$ u(\tau) = \frac{1}{1 + 1.59416\ \mathrm{cn}^2\left[0.537061(\tau + 3.88405); \frac{1}{\sqrt{2}}\right]} \ . \tag{30} $$

The plot of this solution together with its phase portrait is presented in Fig. 3 showing that in this case the bubble does not collapse.

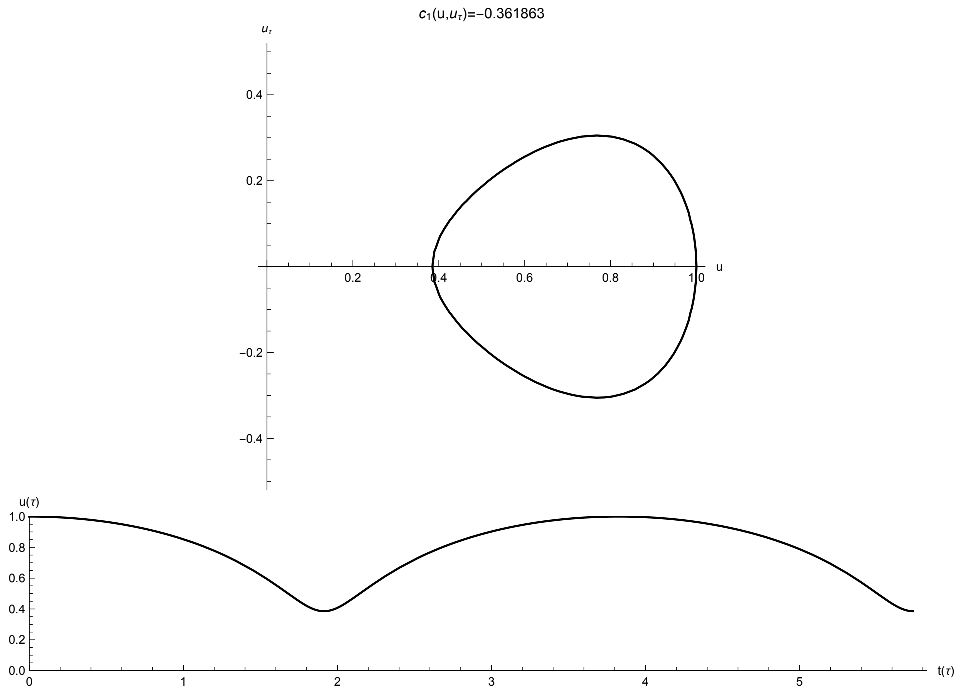

FIG. 3: The phase portrait from (12) and the corresponding periodic parametric solution in terms of Jacobi's elliptic function given by (30).

B. Degenerate cases

We now study the degenerate cases given by $\Delta = 0$ for which (19) becomes

$$ {v_\tau}^2 = \left(-2\alpha + \gamma + \frac{2\xi^2}{3}\right) v^3 + 2\alpha v^2 - \gamma v - \frac{2\xi^2}{3} \ . \tag{31} $$

In this case the discriminant factors as

$$ \Delta = \frac{(6\alpha - 2\xi^2 - 3\gamma)^2(\alpha - \xi^2 - \gamma)^2 \left(16\alpha\xi^2 - 4\xi^4 - 4\xi^2\gamma + 3\gamma^2\right)}{1728} \tag{32} $$

and the solutions given by (21) simplify since the Weierstrass $\wp$ function degenerates into trigonometric or hyperbolic elementary solutions.

i) Trigonometric solutions

There are three possibilities for which $\Delta = 0$.

In the first case, $\alpha = \xi^2 + \gamma$ which is equivalent to $q = 2\left(1 + \frac{\xi^2}{\gamma}\right)$ and implies $c_1 = -\left(\frac{4\xi^2}{3} + \gamma\right)$. Then (31) has a double root at $v = 1$, which can be factored as

$$ {v_\tau}^2 = -\frac{1}{3}(v-1)^2[2\xi^2 + (4\xi^2 + 3\gamma)v] \ . \tag{33} $$

The solution is

$$ v(\tau) = 1 - \frac{3(2\xi^2 + \gamma)}{4\xi^2 + 3\gamma} \sec^2\left[\frac{1}{2}\sqrt{2\xi^2 + \gamma}\,(\tau + \tau_0)\right] \ . \tag{34} $$

However, this case does not satisfy the initial condition $v(0) = 1$, so it will be disregarded as nonphysical.

Secondly, $\alpha = \frac{1}{16}\left(-\frac{3\gamma^2}{\xi^2} + 4\xi^2 + 4\gamma\right)$ which is equivalent to $q = \frac{1}{2} + \frac{\xi^2}{2\gamma} - \frac{3\gamma}{8\xi^2}$, and gives $c_1 = \frac{(2\xi^2+3\gamma)^2}{24\xi^2}$. Then (31) has a simple root for $v = 1$, which can be factored as

$$ {v_\tau}^2 = \frac{1}{24\xi^2}(v-1)[4\xi^2 + (2\xi^2+3\gamma)v]^2 \tag{35} $$

with solution

$$ v(\tau) = -\frac{4\xi^2}{2\xi^2+3\gamma} + \frac{3(2\xi^2+\gamma)}{2\xi^2+3\gamma}\sec^2\left(\frac{\sqrt{2\xi^2+\gamma}\sqrt{2\xi^2+3\gamma}}{4\sqrt{2}\xi}\tau\right) \ , \tag{36} $$

which satisfies the initial condition $v(0) = 1$. The general solution to (12) in parametric form is

$$ \begin{aligned} u(\tau) &= \frac{1}{1+A\tan^2(\theta\tau)} \\ t(\tau) &= -\frac{1}{2\theta(A-1)}\left[\frac{A\tan(\theta\tau)}{1+A\tan^2(\theta\tau)} + \frac{2\theta}{A-1}\tau + \frac{(A-3)\sqrt{A}}{A-1}\tan^{-1}\left(\sqrt{A}\tan(\theta\tau)\right)\right] \ , \end{aligned} \tag{37} $$

where $A = 1 + \frac{4\xi^2}{2\xi^2+3\gamma}$, and $\theta = \frac{\sqrt{2\xi^2+\gamma}\sqrt{2\xi^2+3\gamma}}{4\sqrt{2}\xi}$. Choosing the realistic numerical values $Y = 2.96086 \times 10^8$ Pa, $\sigma = 1.005 \times 10^{-2}$ N/m, and $R_c = 2.94613 \times 10^{-6}$ m, we obtain the values of γ, α, c_1, and q as 0.165966, 0.244479, 0.234769, 2.94613, respectively. With these values, one calculates $A = 2.54136$ and θ =0.38621. The corresponding periodic trigonometric solution (37) and its phase portrait are presented in Fig. 4.

ii) Hyperbolic solutions

This case is found for $\alpha = \frac{1}{6}(2\xi^2 + 3\gamma)$, which is equivalent to $q = 1 + \frac{2\xi^2}{3\gamma}$, and gives $c_1 = 0$. Then (31) is factored as

$$ {v_\tau}^2 = \frac{1}{3}(v-1)\left[2\xi^2 + (2\xi^2+3\gamma)v\right] \ , \tag{38} $$

with solution

$$ v(\tau) = \frac{3\gamma}{2(2\xi^2+3\gamma)} + \frac{(4\xi^2+3\gamma)}{2(2\xi^2+3\gamma)}\cosh\left(\sqrt{\frac{2\xi^2+3\gamma}{3}}\,\tau\right) \tag{39} $$

which satisfies the initial condition $v(0) = 1$. The general solution to (12) in parametric form is

$$ \begin{aligned} u(\tau) &= \frac{1}{1+2B\sinh^2\left(\frac{\tilde\theta\tau}{2}\right)} \\ t(\tau) &= \frac{1}{\tilde\theta(2B-1)}\left[\frac{B\sinh(\tilde\theta\tau)}{1+2B\sinh^2\left(\frac{\tilde\theta\tau}{2}\right)} + \frac{2B-2}{\sqrt{2B-1}}\tan^{-1}\left(\sqrt{2B-1}\tanh\left(\frac{\tilde\theta\tau}{2}\right)\right)\right] \ , \end{aligned} \tag{40} $$

where $B = \frac{1}{2} + \frac{\xi^2}{2\xi^2+3\gamma}$, and $\tilde\theta = \sqrt{\frac{2\xi^2+3\gamma}{3}}$.

Choosing the realistic numerical values $Y = 4.3825 \times 10^8$ Pa, $\sigma = 1.005 \times 10^{-2}$ N/m, and $R_c = 4.3607 \times 10^{-6}$ m, one can obtain the values of γ, α, c_1, and q as 0.165966, 0.361864, 0, 4.3607, respectively. Then $B = 0.885339$, and $\tilde\theta = 0.850722$. For these values, the plot of the hyperbolic solution (40) and its phase portrait are presented in Fig. 4.

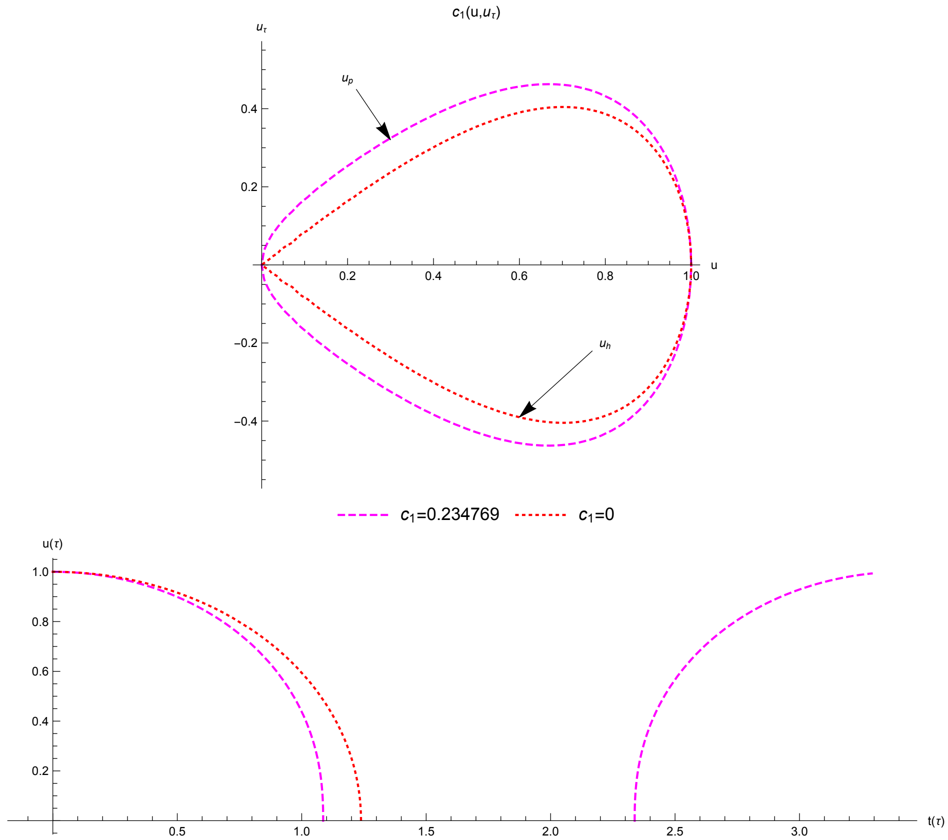

FIG. 4: The phase portraits from (12) and the corresponding periodic trigonometric solution (37) and hyperbolic solution (40).

III. CONCLUSION

In this paper, parametric solutions of the Rayleigh-Plesset equation extended with a term that takes into account the bending pressure due to the elasticity of a shell or capsule surrounding a liquid- or vapor-like substance have been obtained. The general method of Weierstrass elliptic equation using as evolution parameter the Sundman time has been employed. Particular cases that can be important in applications, such as cnoidal and modular-degenerate solutions, are also presented. The simpler, but more particular method using the Abel equation has been briefly described in the appendix. The quotients of the surface and bending pressures and the pressure of the bakground medium together with the Rayleigh collapse time are the other parameters that characterize the solutions displayed in this work.

APPENDIX A: INTEGRATION VIA ABEL'S EQUATION

Proceeding as in [10], the solutions to a general second order ODE of type

$$u_{tt} + f_2(u)u_t + f_3(u) + f_1(u){u_t}^2 + f_0(u){u_t}^3 = 0 \tag{A1}$$

can be obtained via the solutions to Abel's equation of the first kind (and vice-versa)

$$\frac{dy}{du} = f_0(u) + f_1(u)y + f_2(u)y^2 + f_3(u)y^3 \tag{A2}$$

using the substitution

$$u_t = \eta(u(t)) \ , \tag{A3}$$

which turns (A1) into the Abel equation of the second kind in canonical form

$$\eta\eta_t + f_3(u) + f_2(u)\eta + f_1(u)\eta^2 + f_0(u)\eta^3 = 0 \ . \tag{A4}$$

Using the inverse transformation $\eta(u(t)) = 1/y(u(t))$ of the dependent variable, (A4) becomes (A2) and viceversa.

In our case, by comparing (A1) with (5), we identify the nonlinear coefficients to be $f_0(u) = 0$, $f_1(u) = 3/2u$, $f_2(u) = 0$, and $f_3(u) = \xi^2/u + \gamma/u^2 - \alpha/u^3$. Therefore Abel's equation (A2) simplifies to the Bernoulli equation

$$\frac{dy}{du} = f_1(u)y + f_3(u)y^3 \ . \tag{A5}$$

By one quadrature, this equation has the solution

$$y(u) = \pm\frac{u^{3/2}}{\sqrt{c_1 + 2\alpha u - \gamma u^2 - \frac{2\xi^2}{3}u^3}} \ , \tag{A6}$$

and using the inverse transformation $1/y(u(t)) = u_t$ together with (A3), one can obtain (7).

[1] Lord Rayleigh, "VIII. On the pressure developed in a liquid during the collapse of a spherical cavity", Philos. Mag. Ser. 6, **34**, 94 (1917).
[2] M.S. Plesset, "The dynamics of cavitation bubbles", ASME J. Appl. Mech. **16**, 228 (1949).
[3] A. Prosperetti, "Bubbles", Phys. Fluids **16**, 1852 (2004).
[4] A. Malmi-Kakkada and D. Thirumalai, "Generalized Rayleigh-Plesset theory for cell size maintenance in viruses and bacteria", arXiv:1902.07329 (2019).
[5] S.C. Mancas and H.C. Rosu, "Evolution of spherical cavitation bubbles: Parametric and closed-form solutions", Phys. Fluids **28**, 022009 (2016).
[6] N. A. Kudryashov and D. I. Sinelshchikov, Analytical solutions for problems of bubble dynamics, Phys. Lett. A **379**, 798 (2015).
[7] K. Weierstrass, *Mathematische Werke, vol. V* (Johnson, New York, 1915).
[8] E.T. Whittaker and G.N. Watson, *A Course of Modern Analysis* (Cambridge Univesity Press, Cambridge, 1927).
[9] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions: With Formulas, Graphs, and Mathematical Tables* (Courier Dover Publications, New York, 1972).
[10] S.C. Mancas and H.C. Rosu, "Integrable Abel equations and Vein's Abel equation", Math. Meth. Appl. Sci. **39**, 1376-1387 (2016).