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Travelling-Wave Solutions for Korteweg-de Vries-Burgers Equations through Factorizations

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Travelling-wave solutions of the standard and compound form of Korteweg-de Vries-Burgers equations are found using factorizations of the corresponding reduced ordinary differential equations. The procedure leads to solutions of Bernoulli equations of nonlinearity 3/2 and 2 (Riccati), respectively. Introducing the initial conditions through an imaginary phase in the travelling coordinate, we obtain all the solutions previously reported, some of them being corrected here, and showing, at the same time, the presence of interesting details of these solitary waves that have been overlooked before this investigation.

KEY WORDS: travelling wave solutions; factorization method; compound KdVB equation.

1. INTRODUCTION

In this research we will study nonlinear Korteweg-de Vries-Burgers (KdVB) equations which play an important role both in physics and in applied mathematics. First, we will consider the original KdVB equation [1, 2, 3], which can be written as

$$\frac{\partial u}{\partial t} = s\frac{\partial^3 u}{\partial x^3} - \mu\frac{\partial^2 u}{\partial x^2} - \alpha u\frac{\partial u}{\partial x}\ . \tag{1}$$

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Second, we will focus on the so-called compound KdVB equation that has an additional non-linear term [4, 5, 6, 7]

$$\frac{\partial u}{\partial t} = s\frac{\partial^3 u}{\partial x^3} - \mu\frac{\partial^2 u}{\partial x^2} - \alpha u\frac{\partial u}{\partial x} - \beta u^2\frac{\partial u}{\partial x}\ . \tag{2}$$

In both cases the parameters s, μ, α, β are real constants which take into account different effects, as non-linearity, viscosity, turbulence, dispersion or dissipation. As it is well known, these equations have been used as mathematical models for the propagation of waves on elastic tubes [8], and in particular, the KdVB equation has been explicitly derived in the case of ion-acoustic shock waves in multi-electron temperature collisional plasma [9]. These ion-acoustic shock waves have been observed in an unmagnetized plasma [10]. Standard applications are encountered in hydrodynamics and in the theory of diffusive cosmic-ray shocks [11, 12]. Other interesting and more recent applications refer to the dynamics of high-amplitude picosecond strain pulses in sapphire single crystals in which case a KdVB model has been shown to fit the observed behavior at all strains and temperatures under study [13] and explaining the temporal self-shift of small-amplitude dark solitons due to self-induced Raman scattering in optical fibers [14].

We are interested in travelling-wave solutions of the Eqs. (1)–(2), that is, solutions of the form

$$u(x,t) = \phi(\xi), \quad \xi = x - vt, \tag{3}$$

where v is the velocity of the propagation, whose possible values will be determined later. By substituting (3) in (2) and assuming $s \neq 0$, we get an ordinary differential equation (ODE) to be satisfied by $\phi(\xi)$:

$$\frac{d^3\phi}{d\xi^3} - \frac{\mu}{s}\frac{d^2\phi}{d\xi^2} + \left(\frac{v}{s} - \frac{\alpha}{s}\phi - \frac{\beta}{s}\phi^2\right)\frac{d\phi}{d\xi} = 0. \tag{4}$$

A further simplification of this equation is accomplished by means of the following linear transformation of the dependent and independent variables

$$\xi = \frac{s}{\mu}\,\theta, \quad \phi(\xi) = \frac{2\mu^2}{\alpha s}\,w(\theta), \tag{5}$$

which transform Eq. (4) in

$$\frac{d^3 w}{d\theta^3} - \frac{d^2 w}{d\theta^2} + \left(p - 2w - 3q\,w^2\right)\frac{dw}{d\theta} = 0, \tag{6}$$

with

$$p = \frac{vs}{\mu^2}, \quad q = \frac{4\beta\mu^2}{3s\alpha^2}. \tag{7}$$

In the sequel, we will use the simplified form (6), which corresponds to the compound KdVB equation (2) if $q \neq 0$ and to the usual KdVB equation (1) if $q = 0$. It is quite obvious that Eq. (6) admits a first integral

$$\frac{d^2 w}{d\theta^2} - \frac{dw}{d\theta} + \left(p\, w - w^2 - q\, w^3\right) = k, \tag{8}$$

where k is an arbitrary constant. In the following section we will summarize the factorization method to find solutions of general second order differential equations of the type (8). Then, in sections 3 and 4, we will apply this method to KdVB equations.

2. FACTORIZATION OF A SECOND ORDER ODE

The factorization method is a well known technique used to find solutions of ODE. It goes back to some papers of Schrödinger in which he solved some particular examples of the equation bearing his name [15], and it was later developed by Infeld and Hull [16] (for more details, see also [17] and the citations quoted therein).

In a more general context, the factorization of non-linear second order ODE has been previously studied in [18, 19] for equations of the form

$$\frac{d^2 U}{d\theta^2} - \frac{dU}{d\theta} + F(U) = 0, \tag{9}$$

where $F(U)$ is a polynomial function (this is precisely the form of the two KdVB equations (8) considered in the previous section). In these works, Eq. (9) was factorized as

$$[D - f_2(U)][D - f_1(U)]U(\theta) = 0, \tag{10}$$

where $D = d/d\theta$. The following grouping of terms has been proposed in [19]

$$\frac{d^2 U}{d\theta^2} - \left(f_1 + f_2 + \frac{df_1}{dU}U\right)\frac{dU}{d\theta} + f_1 f_2 U = 0, \tag{11}$$

and comparing Eq. (9) with Eq. (11), we get the conditions

$$f_1(U)\, f_2(U) = \frac{F(U)}{U}, \tag{12}$$

$$f_2(U) + \frac{d(f_1(U)\, U)}{dU} = 1. \tag{13}$$

Then, according to [18], any factorization like (10) of an ODE of the form given in Eq. (9), allows to find a compatible first order ODE

$$[D - f_1(U)]U = \frac{dU}{d\theta} - f_1(U)U = 0, \tag{14}$$

whose solution provides a particular solution of (9). In other words, if by some means we are able to find a couple of functions $f_1(U)$ and $f_2(U)$ such that they factorize Eq. (9) in the form (10), solving the ODE (14) we will get particular solutions of (9). The advantage of the factorization presented here is that the two unknown functions $f_1(U)$ and $f_2(U)$ can be found easily by factoring the polynomial expression (12) in terms of linear combinations in rational powers of U.

Now, let us apply this technique of finding particular solutions to the KdVB equations considered in the previous section.

3. FACTORIZATION OF THE KdVB EQUATION

Let us consider first the KdVB equation in the integrated and simplified form given in (8)

$$\frac{d^2 w}{d\theta^2} - \frac{dw}{d\theta} + \left(p\, w - w^2\right) = k. \tag{15}$$

It can be easily proved that this equation can be factorized in the form (10) by means of a linear combination of powers of w only if $k = 0$, which is a very restrictive condition. To circumvent this constraint, we propose a simple displacement on the unknown function

$$w(\theta) = U(\theta) + \delta, \tag{16}$$

where δ is a constant to be determined. With this change, Eq. (15) becomes

$$\frac{d^2 U}{d\theta^2} - \frac{dU}{d\theta} + \left((p - 2\delta)U - U^2\right) = 0, \tag{17}$$

where we have imposed

$$k = p\delta - \delta^2. \tag{18}$$

Now, we can factorize (17) following the procedure exposed in Section 2 with $F(U) = (p - 2\delta)U - U^2$. Taking into account Eqs. (12)–(13), the functions $f_1(U)$ and $f_2(U)$ must be of the form

$$f_1(U) = A\,U^{1/2} + B, \tag{19}$$

$$f_2(U) = (1 - B) - \frac{3}{2}A\,U^{1/2}, \tag{20}$$

with the following values of the two parameters A and B

$$A^2 = \frac{2}{3}, \quad B = \frac{2}{5} \tag{21}$$

together with a constraint between the parameters p and δ, introduced in Eqs. (7) and (16), respectively:

$$p = 2\delta + \frac{6}{25}\ . \tag{22}$$

As we see from (22), the coefficient p (and therefore, according to (7), the velocity v of the traveling wave) depends on the parameter δ, which is arbitrary since it is fixed by the integration constant k in (18).

Another important remark is that if we substitute this value of p in (17), we can realize that we get a kind of *universal equation*, in which all the parameters have disappeared.

Particular solutions of (17) are obtained by solving Eq. (14) in this particular case, which turns out to be the following couple of first order ODE of Bernoulli type:

$$\frac{dU}{d\theta} = \pm\sqrt{\frac{2}{3}}\,U^{3/2} + \frac{2}{5}\,U, \tag{23}$$

whose solutions, for the negative and positive sign, are

$$U^{-}(\theta) = \frac{3}{50}\left[1 + \tanh\left(\frac{\theta - \theta_0}{10}\right)\right]^2, \tag{24}$$

$$U^{+}(\theta) = \frac{3}{50}\left[1 + \coth\left(\frac{\theta - \theta_0}{10}\right)\right]^2. \tag{25}$$

In Figs. 1 and 2 we show a plot of these *universal solutions*.

Returning to the original function $w(\theta)$ in (16) is trivial taking into account (22). Then, using the definitions introduced in Eqs. (3), (5) and (7), we get the following family of regular solutions

$$u^{-}(x,t) = \frac{v}{\alpha} + \frac{3\mu^2}{25\alpha s}\left\{\left[1 + \tanh\left(\frac{\mu(x - vt - \xi_0)}{10s}\right)\right]^2 - 2\right\}, \tag{26}$$

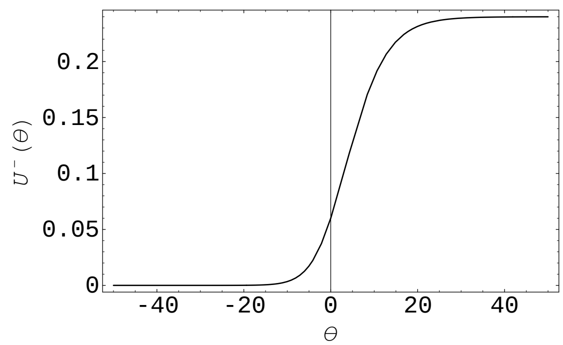

Figure 1: Plot of the *universal regular solution* $U^{-}(\theta)$ given in (24) with $\theta_0 = 0$.

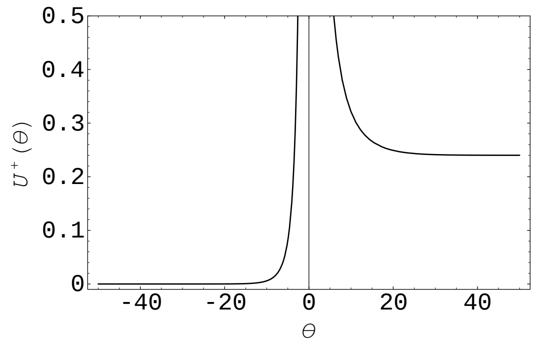

Figure 2: Plot of the *universal singular solution* $U^{+}(\theta)$ given in (25) with $\theta_0 = 0$.

and the following family of singular solutions

$$u^{+}(x,t)=\frac{v}{\alpha}+\frac{3\mu^2}{25\alpha s}\left\{\left[1+\coth\left(\frac{\mu(x-vt-\xi_0)}{10s}\right)\right]^2-2\right\}. \quad (27)$$

Notice that after all these steps we have obtained a couple of families (one regular and one singular) of particular solutions of the KdVB equation (1) depending on two free parameters, which are the arbitrary constant ξ_0 and, what is more important, the velocity v of the travelling wave. In other words, our result indicates that for every value of the velocity we have a family of regular particular solutions of the KdVB equation (1), whose particular features depend on the values of the parameters of the original equation. However, in each case the solutions are always kink-type solutions due to the presence of the *universal solution* $U^{-}(\theta)$ given in (24), which is represented in Fig. 1.

It is interesting to compare our results with others obtained by means of more elaborated or computerized methods. For instance, in [1] the so-called *tanh-method* gives a number of particular solutions called 'regular', 'singular' or 'special'. However, all these solutions are obtained in our result (26) for particular choices of the parameters: their 'regular' solution (20) is obtained when we choose $\mu^2 = 100s^2$, their 'singular' solutions correspond to the elections $\mu^2 = 25s^2$ or $\mu^2 = 100s^2$, and the 'special' solution is found when the velocity of the solitary wave is $v = 6\mu^2/(25s)$. Similarly, in [20, 21, 22], using the 'first integral method', particular solutions are found and reported for specific values of the velocity. From the standpoint of these methods, our solutions (26) and (27), as well as other particular solutions, can be obtained in the standard way that makes use of the elementary symmetries of the KdVB equations such as shifts and Galilean transformations. On the other hand, from the standpoint of the factorization method the equivalent of these symmetries reduces to attributing relative values to the parameters μ and s as stated above.

It is worth noting that if we consider the possibility of having imaginary constants θ_0 in (24)–(25) or ξ_0 in (26)–(27), both types of solutions would be in fact two versions of a unique solution. It is easy to show that such imaginary constants are not forbidden mathematically by the initial conditions. For example, the change $\theta_0 \to \theta_0 + 5i\pi$ in (24) produces (25). More interesting situations are obtained for other complex values, for instance if we choose $\theta_0 = -5i\pi/2$ we get a complex solution of the KdVB equation, whose real and imaginary parts are represented in Figs. 3 and 4, respectively. The behavior of this complex solution, having the periodicity of the trigonometric tangent introduced through the imaginary phase, is quite appealing and is straightforwardly related with the complex scalar solutions mentioned in Refs. [23, 24].

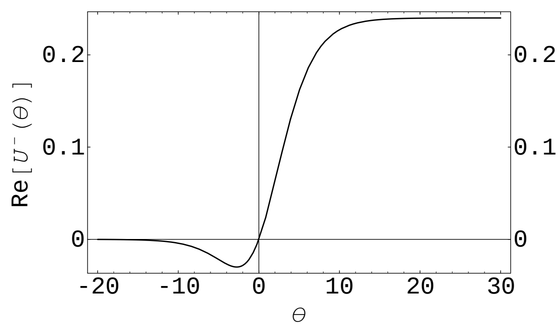

Figure 3: Plot of the real part of the *universal regular solution* $U^{-}(\theta)$ given in (24) with $\theta_0 = -5i\pi/2$.

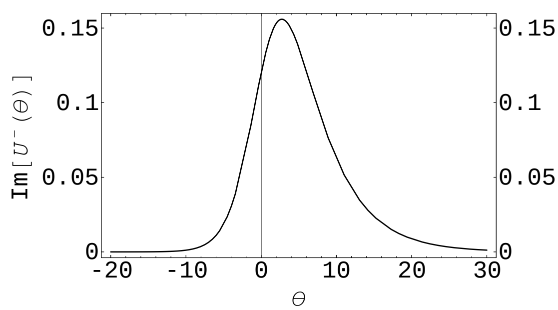

Figure 4: Plot of the imaginary part of the *universal regular solution* $U^{-}(\theta)$ given in (24) with $\theta_0 = -5i\pi/2$.

In fact, if we write $\theta_0 = ia\pi$ (denoting now the solutions as $U^{-}(\theta) \equiv U^{-}(\theta, a)$ to take into account the initial condition), and examine the parametric dependence of the real and imaginary part of the solution, we see that the real part is passing from the pure kink type (for $a = 0$) to the singular solution for $a = -5$, whereas the imaginary part evolves from the pure bell shape to more complex shapes at increasing a, see Figs. 5 and 6, respectively.

Our three-dimensional plots are in agreement with Theorem 1 of Liu and Liu [23] (see also their Remark 3) on complex KdVB solutions that states

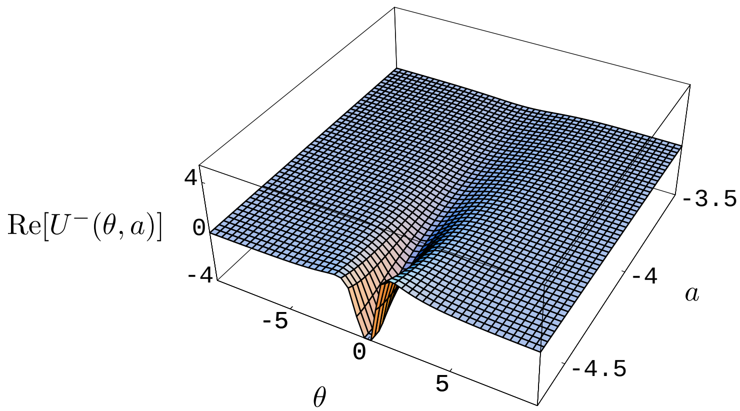

Figure 5: Plot of the real part of the *universal regular solution* $U^-(\theta, a)$ given in (24).

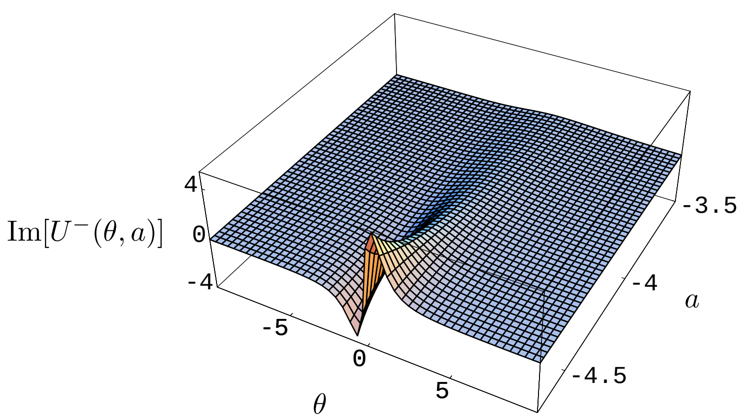

Figure 6: Plot of the imaginary part of the *universal regular solution* $U^-(\theta, a)$ given in (24).

that the real part is kink like whereas the imaginary part is bell-shaped. However, we have found here that this is a consequence of the transition of the regular solution towards the singular one and vice versa when varying the amount of radians in the imaginary phase. In addition, if one looks carefully to the sections of constant a, one will find that there is a range of a in which kink-type solutions with a pocket on the left tail and a bump on the right tail occur. Such kinks have not been previously reported in theory although it seems that they have been observed experimentally [10].

4. FACTORIZATION OF THE COMPOUND KdVB EQUATION

We will undertake the same program for the travelling-wave solutions of the compound KdVB equation (8) with $q \neq 0$. We can factorize it directly as in Eq. (10), taking into account that now $w = U$ and $F(U) = p\,U - U^2 - q\,U^3 - k$, by choosing the factors in the form

$$f_1(U)\,U = A\,U^2 + B\,U + C, \tag{28}$$

$$f_2(U) = 1 - \frac{d(f_1 U)}{dU} = -2A\,U + (1 - B). \tag{29}$$

The restriction conditions (12)–(13) lead us to the following values of the parameters:

$$A^2 = \frac{q}{2}, \qquad B = \frac{A+1}{3A}, \tag{30}$$

$$C = \frac{1}{18}\left(\frac{2-9p}{A} + \frac{1}{A^2} - \frac{1}{A^3}\right) \tag{31}$$

together with

$$\frac{1-2A}{3A}\,C = k. \tag{32}$$

The last relation (32) means that the coefficient C can take any value, since k is an arbitrary integration constant. This fact means that we can get factorizations for any value of p, i.e., for any velocity v of the travelling wave.

Particular solutions of (8) are obtained by the corresponding first order ODE (14), that in our case is

$$\frac{dU}{d\theta} - AU^2 - BU - C = 0 \tag{33}$$

once the parameters A, B, C are replaced by their values (30)–(31). This is a simple Riccati equation whose solutions are well known. One type of solutions is given by

$$U^{\pm}(\theta) = \frac{-1}{3q} \pm \frac{1}{3\sqrt{2q}} \left[1 + \Delta \tanh \left(\frac{\Delta(\theta - \theta_0)}{6} \right) \right] \tag{34}$$

where

$$\Delta = \sqrt{18p + \frac{6}{q} - 3}. \tag{35}$$

Going back to the original variables of Eqs. (3), (5) and (7), the corresponding solutions of the original compound KdVB equation (2) can be written as

$$u^{\pm}(x,t) = -\frac{\alpha}{2\beta} \pm \frac{\mu}{\sqrt{6\beta s}} \left[1 + \Delta \tanh \left(\frac{\mu\Delta(x - vt - \xi_0)}{6s} \right) \right], \tag{36}$$

$$\Delta = \sqrt{\frac{18vs}{\mu^2} + \frac{9s\alpha^2}{2\beta\mu^2} - 3}. \tag{37}$$

A plot of the functions $u^{+}(x,t)$ is given in Fig. 7 for a certain choice of the parameters of the original KdVB equation and different values of the wave velocity.

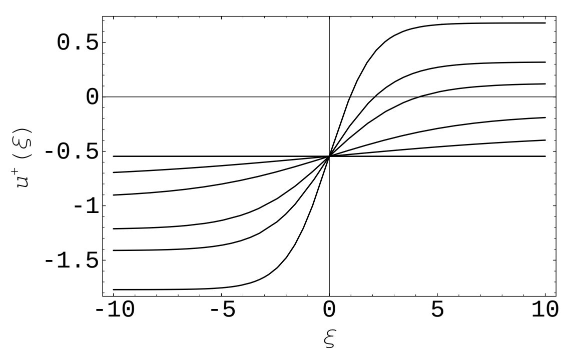

Figure 7: Plot of $u^{+}(x,t) \equiv u^{+}(\xi)$ given in (36) for the following values of the parameters: $\alpha = 3$, $\beta = 2$, $\mu = 1$, $s = 2$, $\xi_0 = 0$ and $v = -1.04, -1.01, -0.94, -0.74, -0.54, -0.04$, being the constant solution the corresponding to $v = -1.04$.

A second type of rational solutions of (33) appears when $\Delta = 0$, that is, for the particular value of v given by

$$6p = 1 - \frac{2}{q} \quad \text{or} \quad v = \frac{\mu^2}{6s} - \frac{\alpha^2}{4\beta}. \tag{38}$$

This singular solution is

$$U_0^{\pm}(\theta) = -\frac{k_0/A}{A + k_0\,\theta} - \frac{A+1}{6A^2}, \tag{39}$$

where $A = \pm\sqrt{q/2}$ and k_0 is an integration constant. For the special case where the free constant $k_0 = 0$, we get a constant solution:

$$U_{0,0}^{\pm}(\theta) = -\frac{A+1}{6A^2}\ . \tag{40}$$

Notice that this constant is directly obtained as a limit of the solutions (34) when $\Delta \to 0$. If $k_0 \neq 0$, the second type solution (40) is recovered taking

$$\theta_0 = -\frac{A}{k_0} - \frac{3\pi i}{\Delta} \tag{41}$$

and then the limit $\Delta \to 0$.

The rational solution (39) expressed in the initial variables becomes

$$u_0^{\pm}(x,t) = -\frac{\alpha}{2\beta}\left(1 \pm \sqrt{\frac{2\beta\mu^2}{3s\alpha^2}}\right) - \frac{6\alpha\mu k_0}{2\beta\mu \pm k_0\sqrt{6s\beta\alpha^2}(x - vt - \xi_0)}\ , \tag{42}$$

where v is given in (38) [3].

The first type solutions (34) were obtained in [6] (with some mistakes, that have been corrected here) using the 'homogeneous balance method', and also in other papers [4, 5, 20, 21, 22] using different methods, but after a considerable amount of computation. However, we have not found any reference to the rational solutions.

[3] If we introduce the parameter $\epsilon = \mu\sqrt{\frac{2\beta}{3s\alpha^2}}$ we can write the rational solution in the simpler form

$$u_0^{\pm}(x,t) = -\frac{\alpha}{2\beta}\left[1 \pm \epsilon + \frac{6\epsilon k_0}{\epsilon \pm k_0(x - vt - \xi_0)}\right]\ ,$$

where $v = \left(\frac{\alpha}{2\beta}\right)^2[\epsilon^2 - 1]$.

5. FINAL REMARKS

In this paper, the factorization method has been shown to be well adapted in the search of particular solutions of non-linear differential equations of KdVB type. This method has a number of advantages compared to others applied to the same problem: i) the basic concept is quite simple and follows the same pattern already used in linear equations; ii) the computations needed to develop it are quite straightforward; iii) it allows to use analytical arguments, for example when looking for the rational solutions of the last section. Thus, the factorization technique is an alternative method to other well-settled procedures and, as we have shown here, can be successfully used to get exact particular KdVB solutions. It can be applied to other non-linear partial differential equations, looking in an efficient way for particular solutions of physical interest. An immediate task is the Kadomtsev-Petviashvili-Burgers (KPB) equation that is currently considered to provide a more realistic framework for the study of non-linear wave phenomena in the low and higher altitude auroral region [25].

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